# Lecture 35

(35.0) Recall: $\mathfrak{g}$ simple Lie algebra over $\mathbb{C}$. $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.

$\forall \lambda \in \mathfrak{h}^* \rightsquigarrow M_\lambda$ Verma Module $\rightsquigarrow L_\lambda$ unique irred. quotient of $M_\lambda$.

**Theorem.** $L_\lambda$ is finite-dim'l $\iff \lambda \in P_+$.

In this case $L_\lambda$ has the following presentation:

generated (as $\mathfrak{g}$-repn.) by $\mathbb{1}_\lambda$

Rel$^{\text{n}}$s:
$$\mathfrak{g}_\alpha \cdot \mathbb{1}_\lambda = 0 \quad \forall \alpha \in R_+$$
$$h \cdot \mathbb{1}_\lambda = \lambda(h)\, \mathbb{1}_\lambda \quad \forall h \in \mathfrak{h}$$
$$f_i^{\lambda(h_i)+1}\, \mathbb{1}_\lambda = 0 \quad \forall i \in \{1, \dots, \ell\}.$$

(35.1) Proof of $\lambda \in P_+ \Rightarrow L_\lambda$ is f.d. and has the presentation given above.

Let $\widetilde{L}_\lambda = M_\lambda \big/ \text{submodule generated by } f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda$

**Lemma.** (1) Let $\widetilde{K}_\lambda \subset M_\lambda$ be the submodule generated by $f_i^{\lambda(h_i)+1}\, \mathbb{1}_\lambda$ ($i \in \{1, \dots, \ell\}$). Then $\widetilde{K}_\lambda$ is a proper submodule.

(2) $\widetilde{L}_\lambda$ is integrable hence finite-dim'l.

Proof. (1) Note $e_i f_i^{\lambda(h_i)+1} \cdot \mathbb{1}_\lambda = 0$ (by $sl_2$ calculation) $\quad e_j f_i^{\lambda(h_i)+1} \mathbb{1}_\lambda = 0 \quad j \neq i$ $\quad (e_j f_i = f_i e_j$ if $j \neq i)$

So by PBW theorem

$$\widetilde{K}_\lambda[\mu] \neq 0 \implies \mu \leq \lambda - (\lambda(h_i)+1)\alpha_i \text{ for some } i \in \{1, \dots, \ell\}$$

hence $\widetilde{K}_\lambda$ is proper.

(2) $\mathrm{ad}(f_i) \circlearrowright \mathfrak{g}$ is nilpotent (by presentation of $\mathfrak{g}$)

For $x_1, \dots, x_n \in \mathfrak{g}$, the following holds in $\mathcal{U}(\mathfrak{g})$

$$\frac{(\mathrm{ad}\, f_i)^p}{p!}(x_1 \dots x_n) = \sum_{p_1 + \dots + p_n = p} \frac{(\mathrm{ad}\, f_i)^{p_1}(x_1)}{p_1!} \cdots \frac{(\mathrm{ad}\, f_i)^{p_n}(x_n)}{p_n!}$$

$\implies \mathrm{ad}(f_i) \circlearrowright \mathcal{U}(\mathfrak{g})$ is locally nilpotent

Now $f_i$ acting on $\mathbb{1}_\lambda \in \widetilde{L}_\lambda$ is nilpotent and $\forall N \geq 0$, $x \in \mathcal{U}(\mathfrak{g})$

$$f_i^N x \mathbb{1}_\lambda = \sum_{k=0}^{N} \binom{N}{k} (\mathrm{ad}\, f_i)^k (x) \cdot f_i^{N-k} \mathbb{1}_\lambda \implies \text{each } f_i$$

acts locally nilpotently on $\widetilde{L}_\lambda$. Hence $\widetilde{L}_\lambda$ is integrable and by Prop (34.4) page 4, $\forall\, w \in W,\ \mu \in \mathfrak{h}^*$ $\quad \widetilde{L}_\lambda[\mu] \xrightarrow{\sim} \widetilde{L}[w\mu]$

By Lemma (19.1) page 1, we can make $w\mu(h_i) \geq 0 \ \forall i$

Also, by Prop (34.2), $P(\tilde{L}_\lambda) \subset \lambda - Q_+$, so we get

$$P(\tilde{L}_\lambda) \subset \bigcup_{\substack{\mu \in P_+ \\ \mu \leq \lambda}} W.\mu \quad \leftarrow \text{finite set}$$

$\Rightarrow \tilde{L}_\lambda$ is f.d. $\square$

(35.2) Cor. $L_\lambda$ is f.d.

Proof. $\tilde{K}_\lambda \subset K_\lambda$ (unique max'l proper submodule of $M_\lambda$)

$$\Rightarrow \underset{M_\lambda/\tilde{K}_\lambda}{\overset{\tilde{L}_\lambda}{\shortparallel}} \longrightarrow\!\!\!\!\rightarrow \underset{M_\lambda/K_\lambda}{\overset{L_\lambda}{\shortparallel}}$$

Hence $\dim L_\lambda < \infty$ $\square$

(35.3) Prop. $\tilde{L}_\lambda = L_\lambda$

Proof. By complete reducibility $\tilde{L}_\lambda = L_\lambda \oplus L'_\lambda$.

Since $L'_\lambda$ is f.d. we can find $0 \neq v \in L'_\lambda[\mu]$ $(\mu \in P_+)$

s.t. $e_\alpha v = 0 \ \forall \alpha \in R_+$ $(\mu \leq \lambda)$

$hv = \mu(h) v \ \forall h \in \mathfrak{h}$

But then $C_\mathfrak{g} v = (|\mu+\delta|^2 - |\delta|^2) v = (|\lambda+\delta|^2 - |\delta|^2) v$

by cor 34.5

Let $\alpha = \lambda - \mu \in Q_+$. Then

$$|\mu+\delta|^2 = |\lambda+\delta|^2 \iff |\mu+\delta|^2 = |\mu+\alpha+\delta|^2$$
$$= |\mu+\delta|^2 + |\alpha|^2 + 2(\mu+\delta, \alpha)$$

i.e. $\underset{\uparrow}{2(\mu+\delta,\alpha)} + \underset{\uparrow}{(\alpha,\alpha)} = 0$

both these are $>0$ contradiction $\square$

(35.4) Examples $B_2$ : $\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$ $\quad \omega_1 = \alpha_1 + \frac{\alpha_2}{2}$

$\omega_2 = \alpha_1 + \alpha_2$

$L_{\omega_1}$ :
(4 dim'l)

$$\begin{tikzcd} & f_1 \mathbb{1}_{\omega_1} \arrow[d] & \mathbb{1}_{\omega_1} \arrow[l] \\ f_1 f_2 f_1 \mathbb{1}_{\omega_1} & f_2 f_1 \mathbb{1}_{\omega_1} \arrow[l] & \end{tikzcd}$$

rel$^n$s: $f_2^2 f_1 - 2 f_2 f_1 f_2 + f_1 f_2^2 = 0$

$f_1^3 f_2 - 3 f_1^2 f_2 f_1 + 3 f_1 f_2 f_1^2 - f_2 f_1^3 = 0$

$f_1^2 \mathbb{1}_{\omega_1} = 0 = f_2 \mathbb{1}_{\omega_1}$

$L_{\omega_2}$ :
(5 dim'l)

$$\begin{tikzcd} & & \mathbb{1}_{\omega_2} \arrow[dr] & \\ & f_1^2 f_2 \mathbb{1}_{\omega_2} \arrow[d] & f_1 f_2 \mathbb{1}_{\omega_2} \arrow[l] & f_2 \mathbb{1}_{\omega_2} \arrow[l] \\ & f_2 f_1^2 f_2 \mathbb{1}_{\omega_2} & & \end{tikzcd}$$