

Algebra II - Mid Term I

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Problem 1

(a) We claim that the covariant functor F is given by:

$$F: \mathcal{D} \longrightarrow F(\mathcal{D}) := \{(d_1, d_2) \in \prod_{i=1,2} \text{Hom}(B_i, \mathcal{D}) : d_1 b_1 = d_2 b_2\}$$

$$\begin{array}{ccc} \mathcal{D} & & F(\mathcal{D}) \ni (d_1, d_2) \\ \downarrow f & \longmapsto & \downarrow \quad \quad \downarrow \\ \mathcal{D}' & & F(\mathcal{D}') \ni (fd_1, fd_2) \end{array}$$

• Check F is well-defined:

$$fd_1 b_1 = f(d_2 b_2) = (fd_2) b_2 \Rightarrow (fd_1, fd_2) \in F(\mathcal{D}')$$

• Check F is a covariant functor:

$$\text{For } \mathcal{D} \xrightarrow{\text{Id}} \mathcal{D}, F(\text{Id}_{\mathcal{D}})(d_1, d_2) = (\text{Id}_{\mathcal{D}} \circ d_1, \text{Id}_{\mathcal{D}} \circ d_2) = (d_1, d_2) \Rightarrow F(\text{Id}_{\mathcal{D}}) = \text{Id}_{F(\mathcal{D})}$$

$$\text{For } \mathcal{D} \xrightarrow{f} \mathcal{D}' \xrightarrow{g} \mathcal{D}'', F(g \circ f)(d_1, d_2) = (gf d_1, gf d_2)$$

$$(F(g) \circ F(f))(d_1, d_2) = F(g)(fd_1, fd_2) = (gf d_1, gf d_2)$$

• Suppose F is representable. Then $\exists C \in \mathcal{C}$ s.t. $F \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(C, -)$.

Therefore,

$$\bullet F(C) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(C, C) \Rightarrow \exists (c_1, c_2) \in F(C), \text{ i.e.}$$

$$B_1 \xrightarrow{c_1} C, B_2 \xrightarrow{c_2} C \text{ s.t. } c_1 b_1 = c_2 b_2$$

$$\bullet \forall D \in \mathcal{C}, F(D) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(C, D) \Rightarrow$$

$$\forall (d_1, d_2) \in F(D), \text{ i.e. } B_1 \xrightarrow{d_1} D, B_2 \xrightarrow{d_2} D \text{ s.t. } d_1 b_1 = d_2 b_2.$$

$$\exists! C \xrightarrow{f} D \text{ s.t. } d_l = f \circ c_l \text{ (for } l=1,2)$$

(b) By the universal property of \tilde{B} , we get $\exists! \pi: \tilde{B} \rightarrow C$ s.t.

$$\pi \circ l_1 = c_1, \pi \circ l_2 = c_2$$

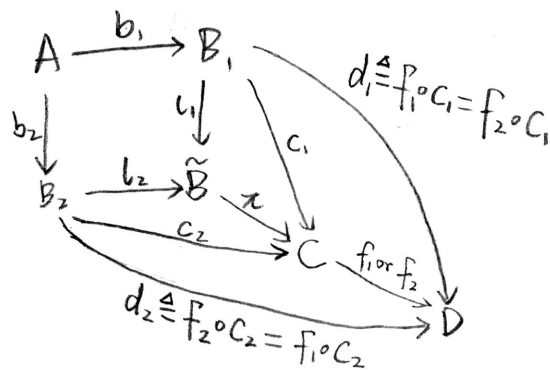
$$\begin{array}{ccc} & B_1 & \\ & \downarrow l_1 & \searrow c_1 \\ B_2 & \xrightarrow{l_2} & \tilde{B} \xrightarrow{\exists!} C \\ & \searrow c_2 & \end{array}$$

Claim: π satisfies the required property.

• Check $\circ \pi$ is one-one.

Suppose $f_1 \circ \pi = f_2 \circ \pi$ for some $f_1, f_2: C \rightarrow D$.

(If f_1, f_2 satisfy the same universal property of C , then $f_1 = f_2$.)



Note that $f_1 \circ C_2 = f_1 \circ (\pi \circ l_2) = f_2 \circ \pi \circ l_2 = f_2 \circ C_2$

Similarly, $f_1 \circ C_1 = f_2 \circ C_1$.

Define $d_1 \triangleq f_1 \circ C_1 = f_2 \circ C_1$, $d_2 \triangleq f_1 \circ C_2 = f_2 \circ C_2$.

Then $d_1 b_1 = (f_1 \circ C_1) \circ b_1 = (f_1 \circ \pi) \circ (l_1 \circ b_1) = (f_2 \circ \pi) \circ (l_2 \circ b_2) = d_2 b_2$

By the u.p. of C , $\exists! f$ s.t. $f \circ C_2 = d_2$, $f \circ C_1 = d_1$

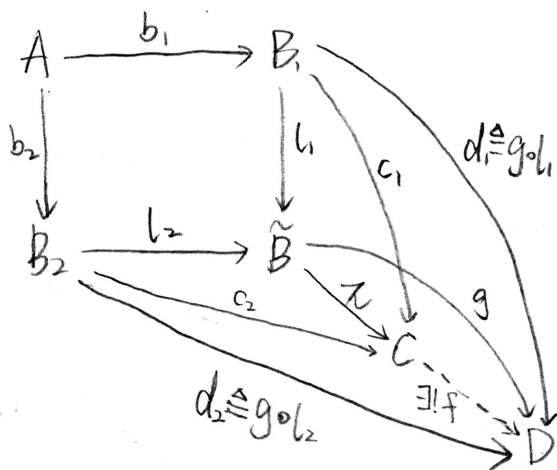
Since f_1, f_2 both satisfy the u.p. of C , $f_1 = f_2$.

• $\forall f \circ \pi \in \text{Im}(\circ \pi)$ with $f \in \text{Hom}_{\mathcal{C}}(C, D)$, we have

$$f \circ \pi \circ l_1 \circ b_1 = f \circ C_1 \circ b_1 = f \circ C_2 \circ b_2 = f \circ \pi \circ l_2 \circ b_2$$

• Suppose $g \in \text{Hom}_{\mathcal{C}}(\tilde{B}, D)$ s.t. $g \circ l_1 b_1 = g \circ l_2 b_2$. We want to show

$g \in \text{Im}(\circ \pi)$, i.e. $\exists f \in \text{Hom}_{\mathcal{C}}(C, D)$ s.t. $g = f \circ \pi$.



Define $d_1 = g \circ l_1$

$d_2 = g \circ l_2$

Then $d_1 b_1 = g \circ l_1 b_1 = g \circ l_2 b_2 = d_2 b_2$

By the u.p. of C , $\exists! f$ s.t.

$f \circ C_1 = d_1$, $f \circ C_2 = d_2$

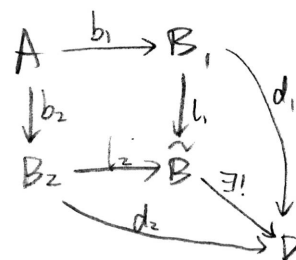
Check $g = f \circ \pi$.

$$f \circ \pi \circ l_1 = f \circ C_1 = d_1 \quad \& \quad g \circ l_1 = d_1$$

$$f \circ \pi \circ l_2 = f \circ C_2 = d_2 \quad \& \quad g \circ l_2 = d_2$$

$\Rightarrow f \circ \pi$ & g both satisfy the u.p. of \tilde{B} :

$\Rightarrow f \circ \pi = g$.



Claim: π is unique. (Page 6 of this answer set)

Problem 2.

proof:

" \Rightarrow " By Theorem 10.1, Page 2.

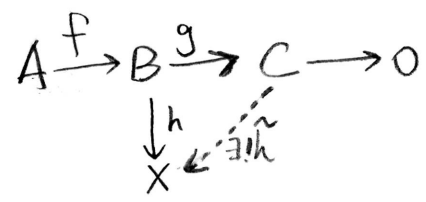
" \Leftarrow " By Remark 9.6, Page 8, it suffices to check $(g, c) = \text{Coker}(A \xrightarrow{f} B)$.

i.e. to prove

① g is surj.

② $\forall B \xrightarrow{h} X$ st. $h \circ f = 0$,

$\exists! \tilde{h}: C \rightarrow X$ st. $\tilde{h} \circ g = h$.



For ①, $\text{Hom}_{\mathcal{C}}(C, X) \xrightarrow{- \circ g} \text{Hom}_{\mathcal{C}}(B, X)$ is inj, and thus one-one in \mathcal{A}_B
 $\Rightarrow g$ is surj.

For ②, $h \circ f = 0 \Rightarrow h \in \text{Ker}(- \circ f) = \text{Im}(- \circ g)$
 $\Rightarrow \exists \tilde{h} \in \text{Hom}_{\mathcal{C}}(C, X)$ st. $h = \tilde{h} \circ g$
 $- \circ g$ is inj $\Rightarrow \tilde{h}$ is unique.

□

Problem 3.

proof:

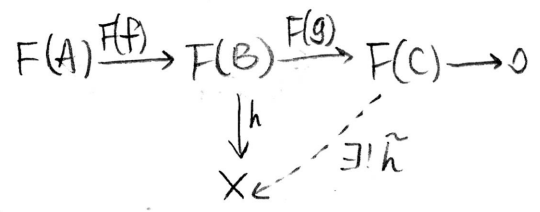
Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be any short exact sequence in \mathcal{A} , we want to prove $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \rightarrow 0$ is exact.

By Remark 9.6, Page 8, it suffices to show $(F(g), F(C)) = \text{Coker}(F(A) \xrightarrow{F(f)} F(B))$

i.e. to prove ① $F(g)$ is surj

② $\forall F(B) \xrightarrow{h} X$ st. $h \circ F(f) = 0$,

$\exists! \tilde{h}: F(C) \rightarrow X$ st. $\tilde{h} \circ F(g) = h$.



For ①, it equals to prove $\text{Hom}_{\mathcal{B}}(F(C), Z) \xrightarrow{- \circ F(g)} \text{Hom}_{\mathcal{B}}(F(B), Z)$ is one-one, $\forall Z \in \mathcal{B}$

F and G adjoint $\Rightarrow \begin{array}{c} |S \\ \text{Hom}_{\mathcal{A}}(C, G(Z)) \xrightarrow{- \circ g} \text{Hom}_{\mathcal{A}}(B, G(Z)) \end{array}$

which is one-one because $g: B \rightarrow C$ is surj.

Therefore, $- \circ F(g)$ is one-one.

For ②, $\text{Hom}_A(B, G(X)) \xleftarrow{\beta_{B,X}^{-1}} \text{Hom}_B(F(B), X)$ (morphisms among $\text{Hom}(\cdot, \cdot)$ are gp hom. So they map 0 to 0.)

$$\begin{array}{ccc} \text{Hom}_A(B, G(X)) & \xleftarrow{\beta_{B,X}^{-1}} & \text{Hom}_B(F(B), X) \\ \downarrow -\circ f & \begin{array}{c} \beta_{B,Y}^{-1}(h) \leftarrow h \\ \downarrow \\ \beta_{B,X}^{-1}(h) \circ f = 0 \end{array} & \downarrow -\circ F(f) \\ \text{Hom}_A(A, G(X)) & \xleftarrow{\beta_{A,X}^{-1}} & \text{Hom}_B(F(A), X) \end{array}$$

By Problem 2, $\beta_{B,X}^{-1}(h) \in \text{Ker}(-\circ f) = \text{Im}(-\circ g)$
 $\Rightarrow \exists j \in \text{Hom}_A(C, G(X))$ s.t. $\beta_{B,X}^{-1}(h) = j \circ g$

Let $\tilde{h} = \beta_{C,X}(j)$. Then

$$\begin{array}{ccc} \text{Hom}_A(C, G(X)) & \xrightarrow{\sim} & \text{Hom}_B(F(C), X) \\ \downarrow -\circ g & & \downarrow -\circ F(g) \\ \text{Hom}_A(B, G(X)) & \xrightarrow{\sim} & \text{Hom}_B(F(B), X) \end{array}, \text{ with}$$

$$\begin{array}{ccc} j & \mapsto & \tilde{h} \\ \downarrow & & \searrow \\ \beta_{B,X}^{-1}(h) & \mapsto & h = \tilde{h} \circ F(g) \end{array}$$

$-\circ F(g)$ one-one $\Rightarrow \tilde{h}$ is unique

So $\exists! \tilde{h}: F(C) \rightarrow X$ s.t. $\tilde{h} \circ F(g) = h$ □

Bonus Problem (Remark: the morphisms are always R -linear, so it is enough to consider pure form $x \otimes y$, instead of general tensors.)

Pf: Write $F = M \otimes_R -$, $G = \text{Hom}_A(M, -)$.

By Prop 4.1, Page 2, it suffices to prove

$$\begin{array}{l} \exists \varepsilon: FG \rightarrow \text{Id}_A \text{ s.t. } F \rightarrow FGF \rightarrow F \text{ are both identity.} \\ \eta: \text{Id}_A \rightarrow GF \quad G \rightarrow GFG \rightarrow G \end{array}$$

$\forall N \in \mathcal{A}$, define $\varepsilon_N: FG(N) = M \otimes_R \text{Hom}_A(M, N) \rightarrow N$ to be a R -linear map s.t.

$$x \otimes f \mapsto f(x)$$

define $\eta_N: N \rightarrow GF(N) = \text{Hom}_A(M, M \otimes_R N)$ to be

$$y \mapsto \left(\begin{array}{l} \eta_N(y): M \rightarrow M \otimes_R N \\ x \mapsto x \otimes y \end{array} \right)$$

The well-definedness is easy to check.

• Check ε is a natural transformation.

Note that $\forall N \xrightarrow{\alpha} N'$, $F(N) \xrightarrow{F(\alpha) = \text{Id} \otimes \alpha} F(N')$

$G(N) \xrightarrow{G(\alpha) = \alpha \circ -} G(N')$

$$\forall N \xrightarrow{\alpha} N', \quad \begin{array}{ccc} M \otimes_R \text{Hom}_A(M, N) & \xrightarrow{\varepsilon_N} & N \\ \text{FG}(\alpha) \downarrow & \text{\textcircled{1}} & \downarrow \alpha \\ M \otimes_R \text{Hom}_A(M, N') & \xrightarrow{\varepsilon_{N'}} & N' \end{array}$$

$$\begin{array}{ccc} x \otimes f & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \otimes (\alpha \circ f) & \longmapsto & \alpha f(x) \end{array}$$

with

So ① commutes

• Check η is a natural transformation

$$\forall N \xrightarrow{\alpha} N', \quad \begin{array}{ccc} N & \xrightarrow{\eta_N} & \text{Hom}_A(M, M \otimes_R N) \\ \alpha \downarrow & \text{\textcircled{2}} & \downarrow \text{GF}(\alpha) \\ N' & \xrightarrow{\eta_{N'}} & \text{Hom}_A(M, M \otimes_R N') \end{array}$$

$$\begin{array}{ccc} y & \longmapsto & \left(\begin{array}{l} \eta_N(y): M \rightarrow M \otimes_R N \\ x \mapsto x \otimes y \end{array} \right) \\ \downarrow & & \downarrow \\ \alpha(y) & \longmapsto & \left(\begin{array}{l} \eta_{N'}(\alpha(y)): M \rightarrow M \otimes_R N' \\ x \mapsto x \otimes \alpha(y) \end{array} \right) \end{array}$$

with

So ② commutes

• $\forall N \in \mathcal{A}$,

$$\begin{array}{c} \text{Id} \otimes \eta_N \\ \parallel \\ F(N) = M \otimes_R N \xrightarrow{F(\eta_N)} M \otimes_R \text{Hom}_A(M, M \otimes_R N) \xrightarrow{\varepsilon_{F(N)}} F(N) = M \otimes_R N \\ x \otimes y \longmapsto x \otimes \eta_N(y) \left(\begin{array}{l} \eta_N(y): M \rightarrow M \otimes_R N \\ x \mapsto x \otimes y \end{array} \right) \longmapsto \eta_N(y)(x) = x \otimes y \end{array}$$

$$\Rightarrow \varepsilon_{F(N)} \circ F(\eta_N) = \text{Id}_{F(N)}$$

$$\begin{array}{c} \text{GF}(\eta_N) \\ \parallel \\ G(N) = \text{Hom}_A(M, N) \xrightarrow{G(\eta_N)} \text{Hom}_A(M, M \otimes_R \text{Hom}_A(M, N)) \xrightarrow{G(\varepsilon_N) = \varepsilon_N \circ -} \text{Hom}_A(M, N) \\ f \longmapsto \left(\begin{array}{l} \eta_{G(N)}(f): M \rightarrow M \otimes_R \text{Hom}_A(M, N) \\ x \mapsto x \otimes f \end{array} \right) \longmapsto (x \mapsto f(x)) \end{array}$$

$$(G(\varepsilon_N) \circ \eta_{G(N)}(f))(x) = (\varepsilon_N \circ \eta_{G(N)}(f))(x) = \varepsilon_N(x \otimes f) = f(x)$$

$$\Rightarrow G(\varepsilon_N) \circ \eta_{G(N)} = \text{Id}_{G(N)}$$

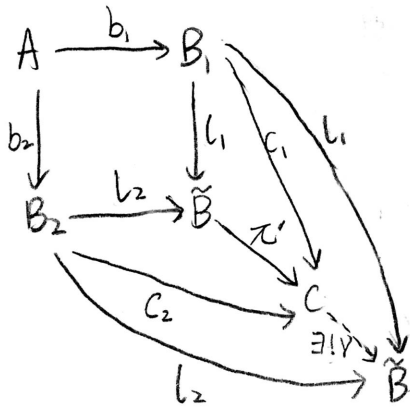
Therefore, F & G are adjoint.

$\frac{35}{35}$ + Bonus

Perfect!

6

Suppose \exists another $\pi': \tilde{B} \rightarrow C$ satisfies the condition.



By u.p. of \tilde{B} , to prove $\pi = \pi'$, it suffices to prove $\pi \circ l_1 = c_1$ and $\pi \circ l_2 = c_2$

By taking $D = C$, we get

$$\text{Id}_C \circ \pi \circ l_1 b_1 = \text{Id}_C \circ \pi \circ l_2 b_2$$

$$\Downarrow$$

$$(\pi l_1) \circ b_1 = (\pi l_2) \circ b_2$$

By taking $D = \hat{B}$, $\exists! \gamma: C \rightarrow \hat{B}$ s.t. $\gamma \circ c_1 = l_1$
 $\gamma \circ c_2 = l_2$

$$\pi \gamma \pi' l_1 b_1 = \pi' \gamma \pi' l_2 b_2 \Rightarrow \pi \gamma \pi' \in \text{Im}(-\circ \pi)$$

$-\circ \pi$ is one-one

$$\text{Id} \pi' l_1 b_1 = \text{Id} \pi' l_2 b_2$$

$$\Rightarrow \text{Id} = \pi' \gamma \Rightarrow \begin{aligned} \pi' l_1 &= \pi' \gamma c_1 = c_1 \\ \pi' l_2 &= \pi' \gamma c_2 = c_2 \end{aligned}$$