Algebra II – Mid Term I

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Problem 1

(a) We claim that the covariant functor F is given by:

$$F : D \longmapsto F(D) := \{(d_1, d_2) \in \prod_{i=1,2} \mathrm{Hom}(B_i, D) : d_1 b_1 = d_2 b_2\}$$

$$\begin{array}{ccc} D & & F(D) \ni (d_1, d_2) \\ \downarrow f & \longmapsto & \downarrow \qquad \updownarrow \\ D' & & F(D') \ni (fd_1, fd_2) \end{array}$$

- Check F is well-defined:

$$fd_1 b_1 = f(d_2 b_2) = (fd_2) b_2 \Rightarrow (fd_1, fd_2) \in F(D')$$

- Check F is a covariant functor:

For $D \xrightarrow{Id} D$, $F(Id_D)(d_1, d_2) = (Id_D \circ d_1, Id_D \circ d_2) = (d_1, d_2) \Rightarrow F(Id_D) = Id_{F(D)}$

For $D \xrightarrow{f} D' \xrightarrow{g} D''$, $F(g \circ f)(d_1, d_2) = (gfd_1, gfd_2)$

$$(F(g) \circ F(f))(d_1, d_2) = F(g)(fd_1, fd_2) = (gfd_1, gfd_2)$$

- Suppose F is representable. Then $\exists\, C \in \mathscr{C}$ s.t. $F \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(C, -)$.

Therefore,

- $F(C) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(C, C) \Rightarrow \exists\, (c_1, c_2) \in F(C)$, i.e.

$$B_1 \xrightarrow{c_1} C,\ B_2 \xrightarrow{c_2} C \text{ s.t. } c_1 b_1 = c_2 b_2$$

- $\forall D \in \mathscr{C}$, $F(D) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(C, D) \Rightarrow$

$\forall (d_1, d_2) \in F(D)$, i.e. $B_1 \xrightarrow{d_1} D$, $B_2 \xrightarrow{d_2} D$ s.t. $d_1 b_1 = d_2 b_2$.

$\exists!\ C \xrightarrow{f} D$ s.t. $d_i = f c_i$ (for $i = 1, 2$)

(b) By the universal property of $\check{B}$, we get $\exists!\ \tau : \check{B} \longrightarrow C$ s.t.

$$\tau \circ l_1 = c_1,\ \tau \circ l_2 = c_2$$

$$\begin{tikzcd} & B_1 \arrow[d, "l_1"] \arrow[ddr, "c_1"] & \\ B_2 \arrow[r, "l_2"] \arrow[drr, bend right, "c_2"'] & \check{B} \arrow[dr, dashed, "\exists!"] & \\ & & C \end{tikzcd}$$

Claim: τ satisfies the required property.

- Check $-\circ\pi$ is one-one.

Suppose $f_1\circ\pi = f_2\circ\pi$ for some $f_1, f_2 : C \to D$.

(If f_1, f_2 satisfy the same universal property of C, then $f_1 = f_2$.)

$$\begin{tikzcd}
A \arrow[r, "b_1"] \arrow[d, "b_2"'] & B_1 \arrow[d, "l_1"] \arrow[ddr, "c_1"] \arrow[dddrr, bend left, "d_1 \triangleq f_1\circ c_1 = f_2\circ c_1"] & & \\
B_2 \arrow[r, "l_2"] \arrow[drr, "c_2"] \arrow[ddrrr, bend right, "d_2 \triangleq f_2\circ c_2 = f_1\circ c_2"'] & \tilde{B} \arrow[dr, "\pi"] & & \\
& & C \arrow[dr, "f_1 \text{ or } f_2"] & \\
& & & D
\end{tikzcd}$$

Note that $f_1\circ c_2 = f_1\circ(\pi\circ l_2) = f_2\circ\pi\circ l_2 = f_2\circ c_2$

Similarly, $f_1\circ c_1 = f_2\circ c_1$.

Define $d_1 \triangleq f_1\circ c_1 = f_2\circ c_1$, $d_2 \triangleq f_1\circ c_2 = f_2\circ c_2$.

Then $d_1 b_1 = (f_1\circ c_1)\circ b_1 = (f_1\circ\pi)\circ(l_1\circ b_1) = (f_2\circ\pi)\circ(l_2\circ b_2) = d_2 b_2$

By the u.p. of C, $\exists!\, f$ s.t. $f\circ c_2 = d_2$, $f\circ c_1 = d_1$

Since f_1, f_2 both satisfy the u.p. of C, $f_1 = f_2$.

- $\forall\, f\circ\pi \in \mathrm{Im}(-\circ\pi)$ with $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$, we have

$$f\circ\underline{\pi\circ l_1}\circ b_1 = f\circ c_1\circ b_1 = f\circ c_2\circ b_2 = f\circ\pi\circ l_2\circ b_2$$

- Suppose $g \in \mathrm{Hom}_{\mathcal{C}}(\tilde{B}, D)$ s.t. $g\circ l_1 b_1 = g\circ l_2 b_2$. We want to show $g \in \mathrm{Im}(-\circ\pi)$, i.e. $\exists\, f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ s.t. $g = f\circ\pi$.

$$\begin{tikzcd}
A \arrow[r, "b_1"] \arrow[d, "b_2"'] & B_1 \arrow[d, "l_1"] \arrow[ddr, "c_1"] \arrow[dddrr, bend left, "d_1 \triangleq g\circ l_1"] & & \\
B_2 \arrow[r, "l_2"] \arrow[drr, "c_2"] \arrow[ddrrr, bend right, "d_2 \triangleq g\circ l_2"'] & \tilde{B} \arrow[dr, "\pi"] \arrow[ddrr, bend left, "g"] & & \\
& & C \arrow[dr, dashed, "\exists! f"] & \\
& & & D
\end{tikzcd}$$

Define $d_1 = g\circ l_1$

$d_2 = g\circ l_2$

Then $d_1 b_1 = g_1 l_1 b_1 = g_2 l_2 b_2 = d_2 b_2$

By the u.p. of C, $\exists!\, f$ s.t.

$f\circ c_1 = d_1$, $f\circ c_2 = d_2$

Check $g = f\circ\pi$.

$f\circ\pi\circ l_1 = f\circ c_1 = d_1$ & $g\circ l_1 = d_1$

$f\circ\pi\circ l_2 = f\circ c_2 = d_2$ $\quad$ $g\circ l_2 = d_2$

$\Rightarrow$ $f\circ\pi$ & g both satisfy the u.p. of $\tilde{B}$:

$$\begin{tikzcd}
A \arrow[r, "b_1"] \arrow[d, "b_2"'] & B_1 \arrow[d, "l_1"] \arrow[ddr, bend left, "d_1"] & \\
B_2 \arrow[r, "l_2"] \arrow[drr, bend right, "d_2"'] & \tilde{B} \arrow[dr, "\exists!"] & \\
& & D
\end{tikzcd}$$

$\Rightarrow$ $f\circ\pi = g$.

Claim: π is unique. (Page 6 of this answer set)

Problem 2.

proof:

"$\Rightarrow$" By Theorem 10.1, Page 2.

"$\Leftarrow$" By Remark 9.6, Page 8, it suffices to check $(g, C) = \mathrm{Coker}(A \xrightarrow{f} B)$.

i.e. to prove

① g is surj.

② $\forall\, B \xrightarrow{h} X$ s.t. $h \circ f = 0$,

$\exists!\, \tilde{h} : C \to X$ s.t. $\tilde{h} \circ g = h$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "h"] & C \arrow[r] \arrow[dl, dashed, "\exists!\tilde{h}"] & 0 \\ & X & & \end{tikzcd}$$

For ①, $\mathrm{Hom}_{\mathcal{C}}(C, X) \xrightarrow{-\circ g} \mathrm{Hom}_{\mathcal{C}}(B, X)$ is inj, and thus one-one in Ab.

$\Rightarrow g$ is surj.

For ②, $h \circ f = 0 \Rightarrow h \in \mathrm{Ker}(-\circ f) = \mathrm{Im}(-\circ g)$

$\Rightarrow \exists\, \tilde{h} \in \mathrm{Hom}_{\mathcal{C}}(C, X)$ s.t. $h = \tilde{h} \circ g$

$-\circ g$ is inj $\Rightarrow \tilde{h}$ is unique.

$\square$.

Problem 3.

proof:

Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be any short exact sequence in $\mathcal{A}$, we want to prove $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact.

By Remark 9.6, Page 8, it suffices to show $(F(g), F(C)) = \mathrm{Coker}(F(A) \xrightarrow{F(f)} F(B))$

i.e. to prove ① $F(g)$ is surj

② $\forall\, F(B) \xrightarrow{h} X$ s.t. $h \circ F(f) = 0$,

$\exists!\, \tilde{h} : F(C) \to X$ s.t. $\tilde{h} \circ F(g) = h$.

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] & F(B) \arrow[r, "F(g)"] \arrow[d, "h"] & F(C) \arrow[r] \arrow[dl, dashed, "\exists!\tilde{h}"] & 0 \\ & X & & \end{tikzcd}$$

For ①, it equals to prove $\mathrm{Hom}_{\mathcal{B}}(F(C), Z) \xrightarrow{-\circ F(g)} \mathrm{Hom}_{\mathcal{B}}(F(B), Z)$ is one-one, $\forall\, Z \in \mathcal{B}$.

F and G adjoint $\Rightarrow$ $\cong$ $\cong$

$\mathrm{Hom}_{\mathcal{A}}(C, G(Z)) \xrightarrow{-\circ g} \mathrm{Hom}_{\mathcal{A}}(B, G(Z))$, which is one-one

because $g : B \to C$ is surj.

Therefore, $-\circ F(g)$ is one-one.

For ②.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(B, G(X)) \arrow[d, "-\circ f"'] & \mathrm{Hom}_{\mathcal{B}}(F(B), X) \arrow[l, "\beta_{B,X}^{-1}"'] \arrow[d, "-\circ F(f)"] \\ \mathrm{Hom}_{\mathcal{A}}(A, G(X)) & \mathrm{Hom}_{\mathcal{B}}(F(A), X) \arrow[l, "\beta_{A,X}^{-1}"] \end{tikzcd}$$

$$\begin{tikzcd} \beta_{B,X}^{-1}(h) \arrow[d, mapsto] & h \arrow[l, mapsto] \arrow[d, mapsto] \\ \beta_{B,X}^{-1}(h)\circ f = 0 & h\circ F(f)=0 \arrow[l, mapsto] \end{tikzcd}$$

(morphisms among $\mathrm{Hom}(,)$ are gp hom. So they map 0 to 0.)

By Problem 2, $\beta_{B,X}^{-1}(h) \in \mathrm{Ker}(-\circ f) = \mathrm{Im}(-\circ g)$

$\Rightarrow \exists\, j \in \mathrm{Hom}_{\mathcal{A}}(C, G(X))$ s.t. $\beta_{B,X}^{-1}(h) = j\circ g$.

Let $\tilde{h} = \beta_{C,X}(j)$. Then

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(C, G(X)) \arrow[r, "\sim"] \arrow[d, "-\circ g"'] & \mathrm{Hom}_{\mathcal{B}}(F(C), X) \arrow[d, "-\circ F(g)"] \\ \mathrm{Hom}_{\mathcal{A}}(B, G(X)) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{B}}(F(B), X) \end{tikzcd}, \text{ with } \begin{tikzcd} j \arrow[r, mapsto] \arrow[d, mapsto] & \tilde{h} \arrow[dr, mapsto] & \\ \beta_{B,X}^{-1}(h) \arrow[r, mapsto] & h & \tilde{h}\circ F(g) \arrow[l, phantom, "//"] \end{tikzcd}$$

$-\circ F(g)$ one-one $\Rightarrow \tilde{h}$ is unique

So $\exists!\, \tilde{h} : F(C) \to X$ s.t. $\tilde{h}\circ F(g) = h$

$\square$

<u>Bonus Problem</u> (Remark: the morphisms are always R-linear, so it is enough to consider pure form $x\otimes y$, instead of general tensors.)

Pf: Write $F = M \underset{R}{\otimes} -$, $G = \mathrm{Hom}_{\mathcal{A}}(M, -)$.

By Prop 4.1, Page 2, it suffices to prove

$\exists\ \varepsilon : FG \longrightarrow \mathrm{Id}_{\mathcal{A}}$ s.t. $F \to FGF \longrightarrow F$ are both identity.

$\eta : \mathrm{Id}_{\mathcal{A}} \longrightarrow GF$ $\qquad G \to GFG \longrightarrow G$

$\forall N \in \mathcal{A}$, define $\varepsilon_N : FG(N) = M \underset{R}{\otimes} \mathrm{Hom}_{\mathcal{A}}(M, N) \longrightarrow N$ to be a R-linear map s.t.

$$x \otimes f \longmapsto f(x)$$

define $\eta_N : N \longrightarrow GF(N) = \mathrm{Hom}_{\mathcal{A}}(M, M \underset{R}{\otimes} N)$ to be

$$y \longmapsto \left(\begin{array}{l} \eta_N(y) : M \longrightarrow M \underset{R}{\otimes} N \\ \qquad x \longmapsto x \otimes y \end{array} \right)$$

The well-definedness is easy to check.

- Check ε is a natural transformation.

Note that $\forall\, N \xrightarrow{\alpha} N'$, $F(N) \xrightarrow{F(\alpha) = \mathrm{Id}\otimes\alpha} F(N')$

$$G(N) \xrightarrow{G(\alpha) = \alpha\circ -} G(N')$$

$$\forall N \xrightarrow{\alpha} N',\quad \begin{tikzcd} M \underset{R}{\otimes} \mathrm{Hom}_{\mathcal{A}}(M,N) \arrow[r, "\varepsilon_N"] \arrow[d, "FG(\alpha)"'] \arrow[dr, phantom, "1"] & N \arrow[d, "\alpha"] \\ M \underset{R}{\otimes} \mathrm{Hom}_{\mathcal{A}}(M,N') \arrow[r, "\varepsilon_{N'}"'] & N' \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} x\otimes f \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x\otimes(\alpha\circ f) \arrow[r, mapsto] & \alpha f(x) \end{tikzcd}$$

So 1 commutes

- Check η is a natural transformation

$$\forall N \xrightarrow{\alpha} N',\quad \begin{tikzcd} N \arrow[r, "\eta_N"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "2"] & \mathrm{Hom}_{\mathcal{A}}(M, M \underset{R}{\otimes} N) \arrow[d, "GF(\alpha)"] \\ N' \arrow[r, "\eta_{N'}"'] & \mathrm{Hom}_{\mathcal{A}}(M, M \underset{R}{\otimes} N') \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} y \arrow[r, mapsto] \arrow[d, mapsto] & \left(\begin{array}{c} \eta_N(y): M \to M \underset{R}{\otimes} N \\ x \mapsto x \otimes y \end{array}\right) \arrow[d, mapsto] \\ \alpha(y) \arrow[r, mapsto] & \left(\begin{array}{c} \eta_{N'}(\alpha(y)): M \to M \underset{R}{\otimes} N' \\ x \mapsto x \otimes \alpha(y) \end{array}\right) \end{tikzcd}$$

So 2 commutes.

- $\forall N \in \mathcal{A}$,

$$F(N) = M \underset{R}{\otimes} N \xrightarrow{\overset{\mathrm{Id}\otimes\eta_N}{=}\, F(\eta_N)} \underbrace{M \underset{R}{\otimes} \mathrm{Hom}_{\mathcal{A}}(M, M \underset{R}{\otimes} N)}_{=\,FGF(N)} \xrightarrow{\varepsilon_{F(N)}} F(N) = M \underset{R}{\otimes} N$$

$$x \otimes y \longmapsto x \otimes \eta_N(y) \left(\begin{array}{c} \eta_N(y): M \to M \underset{R}{\otimes} N \\ x \mapsto x \otimes y \end{array}\right) \longmapsto \eta_N(y)(x) = x \otimes y.$$

$$\Rightarrow \varepsilon_{F(N)} \circ F(\eta_N) = \mathrm{Id}_{F(N)}$$

$$G(N) = \mathrm{Hom}_{\mathcal{A}}(M,N) \xrightarrow{\eta_{G(N)}} \underbrace{\mathrm{Hom}_{\mathcal{A}}(M, M \otimes \mathrm{Hom}_{\mathcal{A}}(M,N))}_{=\,GFG(N)} \xrightarrow{G(\varepsilon_N) = \varepsilon_N \circ -} \mathrm{Hom}_{\mathcal{A}}(M,N)$$

$$f \longmapsto \left(\begin{array}{c} \eta_{G(N)}(f): M \to M \otimes \mathrm{Hom}_{\mathcal{A}}(M,N) \\ x \mapsto x \otimes f \end{array}\right) \longmapsto (x \longmapsto f(x))$$

$$\Uparrow$$

$$\Big(G(\varepsilon_N)(\eta_{G(N)}(f))(x) = (\varepsilon_N \circ \eta_{G(N)}(f))(x) = \varepsilon_N(x \otimes f) = f(x) \Big)$$

$$\Rightarrow G(\varepsilon_N) \circ \eta_{G(N)} = \mathrm{Id}_{G(N)}$$

Therefore, F & G are adjoint.

$\frac{35}{35}$ + Bonus

Perfect!

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Suppose $\exists$ another $\pi' : \tilde{B} \to C$ satisfies the condition.

$$
\begin{tikzcd}
A \arrow[r, "b_1"] \arrow[d, "b_2"'] & B_1 \arrow[d, "l_1"] \arrow[ddr, "c_1"] \arrow[dddr, bend left, "l_1"] & \\
B_2 \arrow[r, "l_2"] \arrow[drr, "c_2"] \arrow[ddrr, bend right, "l_2"'] & \tilde{B} \arrow[dr, "\pi'"] & \\
 & & C \arrow[d, dashed, "\exists ! \gamma"'] \\
 & & \tilde{B}
\end{tikzcd}
$$

By u.p. of $\tilde{B}$, to prove $\pi = \pi'$, it suffices to prove $\pi' \circ l_1 = c_1$ and $\pi' \circ l_2 = c_2$

By taking $D = C$, we get

$$Id_C \circ \pi \circ l_1 b_1 = Id_C \circ \pi \circ l_2 b_2$$

$$\Downarrow$$

$$(\pi' l_1) \circ b_1 = (\pi' l_2) \circ b_2$$

By taking $D = \tilde{B}$, $\exists ! \gamma : C \to B$ s.t. $\gamma \circ c_1 = l_1$

$\gamma \circ c_2 = l_2$

$\pi \gamma \pi' l_1 b_1 = \pi' \gamma \pi' l_2 b_2 \Rightarrow \pi' \gamma \pi' \in Im(- \circ \pi)$

$- \circ \pi$ is one-one

$Id\, \pi' l_1 b_1 = Id\, \pi' l_2 b_2$

$\Rightarrow Id = \pi' \gamma \Rightarrow$ $\pi' l_1 = \pi' \gamma c_1 = c_1$

$\pi' l_2 = \pi' \gamma c_2 = c_2$.