ALGEBRA 2. EXAM 1

Instructions.

- Please read the hypotheses of each problem carefully.
- Please do not seek help from your friends, professors, or the internet!
- You may consult the class notes. If you are going to use a result, that we proved in class, or homework, please write the complete reference (format, for example: Remark 9.6, page 8; or Homework 3, problem 4).
- Please hand in your solutions to Sjuvon during recitation on Tuesday (February 13, 2018) at 11.30AM. We will not accept late solutions, or solutions by email!

Problem 1. Let $\mathcal{C}$ be a category and let A, B_1, B_2 be objects of $\mathcal{C}$ and let $b_\ell : A \to B_\ell$ ($\ell = 1, 2$) be two morphisms in $\mathcal{C}$.

(a) Write the covariant functor $F : \mathcal{C} \to \mathbf{Sets}$, which if representable, is represented by the following data:

An object C of $\mathcal{C}$ together with two morphisms $c_\ell : B_\ell \to C$, $\ell = 1, 2$, such that $c_1 b_1 = c_2 b_2$. This data is then subject to the following (universal) condition:

Given an object D of $\mathcal{C}$ together with two morphisms $d_\ell : B_\ell \to D$, $\ell = 1, 2$, such that $d_1 b_1 = d_2 b_2$, there exists a unique morphism $f : C \to D$ such that $d_\ell = f c_\ell$ (again, $\ell = 1, 2$).

$$\begin{tikzcd}
A \arrow[r, "b_1"] \arrow[d, "b_2"'] & B_1 \arrow[d, "c_1"] \arrow[ddr, bend left, "d_1"] & \\
B_2 \arrow[r, "c_2"'] \arrow[drr, bend right, "d_2"'] & C \arrow[dr, dashed, "f"] & \\
 & & D
\end{tikzcd}$$

(b) Assume that such an object C together with morphisms c_ℓ exists, and assume that $\widetilde{B} := B_1 \oplus B_2$ also exists in $\mathcal{C}$, and let $\iota_\ell : B_\ell \to \widetilde{B}$ be the morphisms from the definition of the direct sum.

Prove that there exists a unique morphism $\pi : \widetilde{B} \to C$ such that the following sequence is *left exact* (see what it means below), for every object D of $\mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(C, D) \xrightarrow{-\circ\pi} \mathrm{Hom}_{\mathcal{C}}(\widetilde{B}, D) \underset{-\circ\iota_2 b_2}{\overset{-\circ\iota_1 b_1}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{C}}(A, D)$$

Meaning. The leftmost map is one to one; and its image is equal to the set of elements of the middle Hom set which go to the same element via the two rightmost maps.

Problem 2. Let $\mathcal{C}$ be an abelian category and consider a sequence of morphisms in $\mathcal{C}$:

$$A \to B \to C \to 0$$

Prove that this sequence is exact if, and only if, for every $X \in \mathcal{C}$ the following sequence of abelian groups is exact:

$$0 \to \mathrm{Hom}_{\mathcal{C}}(C, X) \to \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

Problem 3. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Assume we are given two additive covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that (F, G) is an adjoint pair. Prove that F is always right exact. (Also, G is always left exact, but you don't have to prove it).

Bonus problem. Take R to be a commutative ring with $1 \neq 0$. Let $\mathcal{A}$ be the category of R–modules. For each $M \in \mathcal{A}$, prove that we have a pair of adjoint functors $(M \otimes_R -, \mathrm{Hom}_{\mathcal{A}}(M, -))$ from $\mathcal{A}$ to itself. *Hence, by Problem 2, we get another proof that $M \otimes_R -$ is a right exact functor.*