$\frac{30}{30}$

Bhawesh Mishra

① Flat $\Rightarrow$ Torsion free :

let $0 \neq r \in R$ then map $R \xrightarrow{\mu_r} R$ is
$s \longmapsto sr = r \cdot s$

one-one because R is a domain.

$\because M$ is flat,

$$\begin{array}{ccc} R \otimes_R M & \longrightarrow & R \otimes_R M \\ \wr\| & & \wr\| \\ M & \xrightarrow{\mu_r} & M \\ m & \longmapsto & r \cdot m \end{array}$$

is still injective ie. M is torsion free by def^n.

(Torsion free in PID 'R') $\Rightarrow$ Flat

we will use the fact that M is a flat R-module iff $\mathfrak{a} \otimes N \cong \mathfrak{a}N$ $\forall \mathfrak{a} \triangleleft R$

let $\mathfrak{a} \triangleleft R$ then $\because$ R is a PID,
$$\mathfrak{a} \neq 0$$

$\exists\, 0 \neq a \in \mathfrak{a}$ s.t. $\mathfrak{a} = (a)$. Then we have map

$$\mathfrak{a} \otimes M \xrightarrow{\varphi} \mathfrak{a} M$$
$$(a \otimes m) \longmapsto a \cdot m, \quad \forall a \in \mathfrak{a},\ m \in M$$

which is always a surjective module homomorphism.

& $\operatorname{Ker} \varphi = \{ a \otimes m : a \cdot m = 0 \} = \{ a \otimes n : a = 0 \text{ or } m =$

$\because$ M is torsion-free

So, $\operatorname{Ker} \varphi = (0)$

ie. $(\mathfrak{a} \otimes M \simeq \mathfrak{a} M)$ ie. M is flat

② $R = \mathbb{Z}/27\mathbb{Z}$, $M = N = \mathbb{Z}/9\mathbb{Z}$

Take projective resolution of M as R-module

$$\cdots \longrightarrow \mathbb{Z}/27\mathbb{Z} \xrightarrow{\mu_3} \mathbb{Z}/27\mathbb{Z} \xrightarrow{\mu_9} \mathbb{Z}/27\mathbb{Z} \longrightarrow 0$$

$\Big\downarrow$ Take $\mathrm{Hom}_R(-, N)$ which is $\cong N$ because $-$ spots are all R

$$0 \longrightarrow \mathbb{Z}/9\mathbb{Z} \xrightarrow{\mu_9} \mathbb{Z}/9\mathbb{Z} \xrightarrow{\mu_3} \mathbb{Z}/9\mathbb{Z} \longrightarrow \cdots$$

& $$\mathrm{Ext}^1(M, N) = \frac{\ker\left(\mathbb{Z}/9\mathbb{Z} \xrightarrow{\mu_3} \mathbb{Z}/9\mathbb{Z}\right)}{\mathrm{Im}\left(\mathbb{Z}/9\mathbb{Z} \xrightarrow{\mu_9} \mathbb{Z}/9\mathbb{Z}\right)}$$

$$= \mathbb{Z}/3\mathbb{Z}$$

So, Cardinality of $\mathrm{Ext}^1(M, N) = 3$ //

3) let M be divisible & torsion free over R an I.D.

we use Baer's criterion for injectivity to show M is injective:

- M is an injective R-module iff $\forall \mathfrak{a} \lhd R$ and $f: \mathfrak{a} \longrightarrow M$, $\exists \tilde{f}: R \longrightarrow M$ s.t. $\tilde{f}|_{\mathfrak{a}} = f$.

(Baer's criterion is $\cong$ to $\exists\, m \in M$ s.t. $\forall b \in \mathfrak{a}$, $f(b) = b \cdot M$

we use this version:)

let $\mathfrak{a} \lhd R$ and $f: \mathfrak{a} \longrightarrow M$.

If $\mathfrak{a} = (0)$, statement holds vacuosly.

If $\mathfrak{a} \neq (0)$, pick $0 \neq a \in \mathfrak{a}$.

$\because$ M is divisible, $\exists m \in M$ s.t.

$$f(a) = a \cdot m$$

This m is unique because $f(a) = a \cdot m = a \cdot m'$ gives $a \cdot (m - m') = 0$

$\rightarrow m = m'$ because M is torsion-free.

Now if $b \in \mathfrak{a}$ then

$$a \cdot (b \cdot m) = b(a \cdot m) = b \cdot f(a) = f(ba) = f(ab)$$
$$= a \cdot f(b)$$

$$\rightarrow a[b \cdot m - f(b)] = 0$$

$\rightarrow b \cdot m - f(b) = 0$, since M is torsion-free

So, $f(b) = b \cdot m, \ \forall b \in \mathfrak{a}$

So, M is injective (By Baer's Criterion)