30/30 !

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Algebra 2
Homework 1

(2) $F : \mathcal{C} \to \mathcal{D}$ functor between categories $\mathcal{C}, \mathcal{D}$.

If F is faithful, show $\forall f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, $F(f)$ injective (resp. surjective) $\Rightarrow$ so is f.

Proof: Let $F(f)$ be injective; suppose $f \circ g_1 = f \circ g_2$, $g_i \in \mathrm{Hom}_{\mathcal{C}}(Z,A)$.

Since F is a functor $F(f) \circ F(g_1) = F(f \circ g_1) = F(f \circ g_2) = F(f) \circ F(g_2)$.

$F(f)$ injective $\Rightarrow F(g_1) = F(g_2)$.

If $F(f)$ is surjective, we take $g_1 \circ f = g_2 \circ f$, $g_i \in \mathrm{Hom}_{\mathcal{C}}(B,D)$.

$\Rightarrow F(g_1) \circ F(f) = F(g_2) \circ F(f)$ as above. So $F(g_1) = F(g_2)$.

In either case, F is faithful $\Rightarrow g_1 = g_2$.

Hence, f is injective (resp. surjective).

(9) Let G be a group, G-Sets the category whose objects are sets with a G action, and morphisms s.t.

$f(g \cdot x) = g \cdot f(x)$.

Prove that $\eta : F \to F$ natural isomorphisms of the forgetful functor $F : G\text{-}\underline{Sets} \to \underline{Sets}$ satisfy: $\mathrm{Aut}(F) = G$.

Proof: ($\supseteq$) Let $g \in G$. For all G-sets X, define $\eta_X^{(g)}(t) = g \cdot t$.

~~A G-action is a bijection $X \to X$, and $\eta_X^{(g)}(\sigma \cdot t) = (g \cdot \sigma) \cdot t$ $= (g\sigma) \cdot \eta$~~

Given any $f \in \mathrm{Hom}_{G\text{-}\underline{\text{Sets}}}(X,Y)$, notice that $f(\eta_X^{(g)}(t))$
$= f(g\cdot t) = g\cdot f(t) = \eta_Y^{(g)}(f(t))$. Thus, $\eta^{(g)}$ is natural.

$\eta_X^{(g)}$ is a function $X \to X$, so certainly $\eta_X^{(g)} \in \mathrm{Hom}_{\underline{\text{Sets}}}(F(X), F(X))$.
In addition, $[\eta_X^{(g)}]^{-1} = \eta_X^{(g^{-1})}$, so every $\eta_X^{(g)}$ is, in fact, an isomorphism on $\underline{\text{Sets}}$. In conclusion, $\eta^{(g)}$ is a natural isomorphism $F \longrightarrow F$.

($\subseteq$) Let $\eta \in \mathrm{Aut}(F)$. Denote by $\underline{G}$ the G-set corresponding to G acting on itself by left multiplication.
Take $X \in G\text{-}\underline{\text{Sets}}$ and fix $x \in X$. Define $f_x \in \mathrm{Hom}_{G\text{-}\underline{\text{Sets}}}(\underline{G}, X)$ by $f_x(g) = g\cdot x$. Notice that $f_x(g'\cdot g) = (g'g)\cdot x$
$= g'\cdot(g\cdot x) = g'\cdot f_x(g)$; so f_x commutes with the G action. In this setting, consider the diagram:
(Where $G = F(\underline{G})$ and $f_x = F(f_x)$).

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[d, "f_x"'] & G \arrow[d, "f_x"] \\ X \arrow[r, "\eta_X"'] & X \end{tikzcd}$$

By commutativity of the diagram:
$$\eta_X(x) = \eta_X(e\cdot x) = \eta_X(f_x(e)) = f_x(\eta_G(e)) = \eta_G(e)\cdot x.$$

Denoting $\sigma = \eta_G(e)$, this shows that $\forall X \in G\text{-}\underline{\text{Sets}}$ and all $x \in X$, $\eta_X(x) = \sigma\cdot x$. Hence, $\eta = \eta^{(\sigma)}$ and $\mathrm{Aut}(F) \subseteq G$.

$\square$

(10) Recall the functor $F : \underline{\mathrm{Mat}_k} \to \underline{\mathrm{Vect}_k^{fd}}$

$$n \longmapsto k^n$$

$$\mathrm{Hom}_{\mathcal{D}}(k^n, k^m) = M_{m\times n}(k).$$

Denote $\mathcal{C} = \underline{\mathrm{Mat}}_k$, $\mathcal{D} = \underline{\mathrm{Vect}}_k^{fd}$.

(1) Prove that F is an equivalence of categories.

Proof: Every linear map $T \in \mathrm{Hom}_{\mathcal{D}}(k^n, k^m)$ can be represented as a matrix, taking the OK– standard too canonical bases of k^n, k^m. This means that F is full. It is faithful because the representation is unique. F is also dense because $\forall V \in \mathrm{Vect}_k^{fd}$, if $\dim V = n \Rightarrow V \simeq k^n$.

Thus, F is an equivalence of categories. □

(2) Prove that constructing a functor $G : \underline{\mathrm{Vect}_k^{fd}} \to \underline{\mathrm{Mat}_k}$ together with natural iso's $\varphi : \mathrm{Id}_{\mathrm{Mat}_k} \to GF$

$$\Psi : \mathrm{Id}_{\mathrm{Vect}_k^{fd}} \to FG$$

is the same as making a choice of basis $B_G(V)$ $\forall V \in \mathcal{C}$.

Proof: Take G, φ, Ψ as above. Fix $V \in \mathcal{C}$ with $\dim(V) = n$ so that $FG(V) = k^n$. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in k^n = FG(V)$.
$\uparrow$ ith entry

$\{e_1, \dots, e_n\}$ is a basis for k^n, so $v_i = \Psi_V^{-1}(e_i)$ is a basis for V since $\Psi_V : V \to k^n$ is an isomorphism of vector spaces.

In this setting, for any $T : V \to V$, $FG(T)$ is its associated matrix in the basis $\{v_1, \dots, v_n\}$ because $T = \Psi_V^{-1} \circ FG(T) \circ \Psi_V$.

Ψ_V writes any $v \in V$ in terms of the basis, $FG(T)$ is a matrix that multiplies said vector and Ψ_V^{-1} returns it to V.

Notice your argument makes no mention of φ. This is because φ isn't uniquely determined! i.e., The problem

Conversely, given a basis $B_G(V) = \{v_1, \dots, v_n\}$ for every V, we will construct G, Ψ, φ. Notice that bases for V, W also gives a basis for $\mathrm{Hom}_{\mathcal{E}}(V,W)$: the elements $T_{ij} = \begin{cases} w_j & \text{for } v_i \\ 0 & \text{for } v_k, k \neq i \end{cases}$ defined on $\{v_1, \dots, v_n\}$ and their linear extensions to all V. The associated matrices for the T_{ij} are E_{ij}, so we define G to send V to $\dim(V)$ and $T \in \mathrm{Hom}_{\mathcal{E}}(V,W)$ to $\sum a_{ij} E_{ij}$ if $T = \sum a_{ij} T_{ij}$.

G is a functor because composition of linear maps corresponds to multiplication of their associated matrices and $Id_V = \sum T_{ii}$ is maped to $Id_n = \sum E_{ii}$.

We also define Ψ_V on the basis: $\Psi_V(v_i) = e_i$ and extend it linearly. this is natural because of the properties of matrices associated to linear maps:

$$\begin{tikzcd} V \arrow[r, "\Psi_V"] \arrow[d, "T"'] & K^n \arrow[d, "FG(T)"] \\ W \arrow[r, "\Psi_W"'] & K^m \end{tikzcd}$$

commutes.

$$\boxed{T = \Psi_W^{-1} \circ FG(T) \circ \Psi_V.} \Rightarrow$$

Lastly, φ is already determined. φ_n should be an isomorphism $n \to n$ that makes this diagram commute:

$$\begin{tikzcd} n \arrow[r, "\varphi_n"] \arrow[d, "T"'] & GF(n) \arrow[d, "GF(T)"] \\ m \arrow[r, "\varphi_m"'] & GF(m) \end{tikzcd}$$

F doesn't change $M_{m \times n}(K)$ when it translates it into $\underline{Vect}_K^{fd}$.

So we can lift this diagram to:

$$\begin{tikzcd} F(n) \arrow[r, "\Psi_{K^n}"] \arrow[d, "F(T)"'] & FGF(n) \arrow[d, "FGF(T)"] \\ F(m) \arrow[r, "\Psi_{K^m}"'] & FGF(m) \end{tikzcd}$$

This diagram already commutes by properties of Ψ_{K^n}, so we simply define the action of φ on $\mathrm{Hom}_{\mathcal{D}}(n,m)$ to be that of Ψ on $\mathrm{Hom}_{\mathcal{E}}(K^n, K^m)$.

(3) Given two functors G_1, G_2 obtained by choosing $B_1(V), B_2(V)$ bases for V, prove that the change of basis matrix provides a natural iso. between G_1, G_2.

Proof: Let M_V be the change of basis matrix between $B_1(V)$ and $B_2(V)$. M_V is invertible, and if T is a linear transformation $T: V \to W$ with associated matrices A_1 in bases $B_1(V), B_1(W)$ and A_2 in bases $B_2(V), B_2(W)$, then $A_2 = M_W \cdot A_1 \cdot M_V^{-1}$. This is the same as saying that the diagram on the left

$$\begin{tikzcd} G_1(V) \arrow[r, "M_V"] \arrow[d, "A_1"'] & G_2(V) \arrow[d, "A_2"] \\ G_1(W) \arrow[r, "M_W"'] & G_2(W) \end{tikzcd}$$

commutes. Since $A_1 = G_1(T)$ and $A_2 = G_2(T)$, this means that the ~~natural~~ transformation $\varphi: G_1 \to G_2$ defined by $\varphi_V = M_V$ is natural. Moreover, since M_V, M_W are invertible, φ is an isomorphism. $\square$