ALGEBRA 2 HOMEWORK 1

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Problem 2. Let **Sets** be the category of sets. Given two sets $X, Y \in \mathbf{Sets}$, define a functor $F : \mathbf{Sets} \to \mathbf{Sets}$ as follows: $F(Z) = \mathrm{Hom}_{\mathbf{Sets}}(X, Z) \times \mathrm{Hom}_{\mathbf{Sets}}(Y, Z)$ for every $Z \in \mathbf{Sets}$. For every morphism $f : Z \to Z'$ in **Sets**, $F(f)$ is componentwise composition with f. That is, $F(f)\colon (g_1, g_2) \mapsto (f \circ g_1, f \circ g_2)$. Is this functor representable?

Solution: Yes, F is representable by $X \sqcup Y$, via the map $\eta_Z(g) = (g|_X, g|_Y)$ for all $Z \in \mathbf{Sets}$ and $g \in \mathrm{Hom}_{\mathbf{Sets}}(X \sqcup Y, Z)$. That $\eta_Z : \mathrm{Hom}_{\mathbf{Sets}}(X \sqcup Y, Z) \to \mathrm{Hom}_{\mathbf{Sets}}(X, Z) \times \mathrm{Hom}_{\mathbf{Sets}}(Y, Z)$ is an isomorphism for all Z is clear, since maps on the disjoint union can just be thought of as pairs of maps on each component separately. All that remains to check is that η is natural, i.e. the following diagram commutes for all Z, Z', and $f : Z \to Z'$:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Sets}}(X \sqcup Y, Z) & \xrightarrow{\;f\circ -\;} & \mathrm{Hom}_{\mathbf{Sets}}(X \sqcup Y, Z') \\
\big\downarrow{\scriptstyle (-|_X, -|_Y)} & & \big\downarrow{\scriptstyle (-|_X, -|_Y)} \\
\mathrm{Hom}_{\mathbf{Sets}}(X, Z) \times \mathrm{Hom}_{\mathbf{Sets}}(Y, Z) & \xrightarrow{(f\circ -, f\circ -)} & \mathrm{Hom}_{\mathbf{Sets}}(X, Z') \times \mathrm{Hom}_{\mathbf{Sets}}(Y, Z')
\end{array}$$

For $g : X \sqcup Y \to Z$, this amounts to verifying that

$$(f \circ (g|_X), f \circ (g|_Y)) = ((f \circ g)|_X, (f \circ g)|_Y)$$

as pairs of set maps on X and Y respectively. This is clearly true, so η is indeed a natural isomorphism and thus F is representable.

Problem 6. Let $\mathcal{C}$ be a category and let $\mathbf{Func}\,(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ be the category of contravariant functors from $\mathcal{C}$ to **Sets**. Recall that for $X \in \mathcal{C}$, we defined $\mathsf{h}_X := \mathrm{Hom}_{\mathcal{C}}(-, X) \in \mathbf{Func}\,(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$. Prove that for every $F \in \mathbf{Func}\,(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ we have a bijection between $\mathrm{Hom}_{\mathbf{Func}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}(\mathsf{h}_X, F)$ and $F(X)$.

Solution: Morphisms in $\mathbf{Func}\,(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ are natural transformations of functors from $\mathcal{C}$ to **Sets** so we need to verify that there is a bijection between the set of natural transformations $\mathsf{h}_X \to F$ and elements of the set $F(X)$. First we will verify that every $x_0 \in F(X)$ defines a natural isomorphism. For a fixed $x_0 \in F(X)$, define $\eta^{x_0} : \mathsf{h}_X \to F$ by $\eta^{x_0}_Y(g) = F(g)(x_0)$ for all $Y \in \mathcal{C}$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$. That this is a natural transformation is equivalent to the commutativity of the following diagram for each $Y, Z \in \mathcal{C}$ and $f : Y \to Z$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(Y, X) & \xleftarrow{\;-\circ f\;} & \mathrm{Hom}_{\mathcal{C}}(Z, X) \\
{\scriptstyle \eta^{x_0}_Y}\big\downarrow & & \big\downarrow{\scriptstyle \eta^{x_0}_Z} \\
F(Y) & \xleftarrow{\;F(f)\;} & F(Z)
\end{array}$$

Using the η^{x_0} defined above, this says that we need that for each $g \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$,

$$F(g \circ f)(x_0) = F(f) \circ F(g)(x_0).$$

F was assumed to be a contravariant functor so the above equation holds and η^{x_0} does indeed define a natural transformation.

Now suppose $\eta : \mathsf{h}_X \to F$ is a natural transformation, and define $x_0 := \eta_X(\mathrm{id}_X)$. We will verify that $\eta_Y : \mathrm{Hom}_C(Y, X) \to F(Y)$ is a map of the above form, i.e. that for each $g \in \mathrm{Hom}_C(Y, X)$ we have $\eta_Y(g) = F(g)(x_0)$. This follows easily from the commutativity of the following diagram (which follows from the assumption that η is a natural transformation). In the case that $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is empty there is nothing to check, η_Y is (vacuously) of the desired form. Otherwise let $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, where we have

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(Y, X) & \xleftarrow{-\circ g} & \mathrm{Hom}_{\mathcal{C}}(X, X) \\ {\scriptstyle \eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_X} \\ F(Y) & \xleftarrow{\quad F(g) \quad} & F(X) \end{array}$$

Following id_X yields

$$\eta_Y(g) = F(g)(\eta_X(\mathrm{id}_X)) = F(g)(x_0)$$

as desired. Thus the above construction provides the desired bijection.

Problem 7. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. That is, F is faithful, full and essentially surjective. Prove that F admits both a left and a right adjoint.

Solution: Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors giving an equivalence of categories and let $\psi : \mathrm{id}_{\mathcal{D}} \to FG$ be a natural isomorphism. We will show that (F, G) is an adjoint pair. By reversing F and G and using the other natural isomorphism the same argument shows that (G, F) is an adjoint pair, so this one direction suffices.

For $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we need a map $\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \to \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$. We can accomplish this by defining $\beta_{X,Y}$ to be the composition of the maps

$$\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \xrightarrow{\ F\ } \mathrm{Hom}(F(X), FG(Y)) \xrightarrow{\psi_Y^{-1}\circ -} \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$$

so for each for all $g : X \to G(Y)$, we have $\beta_{X,Y}(g) = \psi_Y^{-1} \circ F(g)$. F is a bijection on Hom sets (since it is faithful and full) and postcomposing with ψ_Y^{-1} is a bijection (its inverse is postcomposing with ψ_Y) so $\beta_{X,Y}$ is also a bijection. It just remains to verify that β is natural in both X and Y, i.e. that for all $X_1, X_2 \in \mathcal{C}$, $Y_1, Y_2 \in \mathcal{D}$, $f : X_1 \to X_2$, and $g : Y_2 \to Y_1$ the following diagram commutes

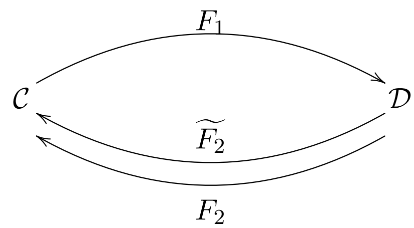
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X_1, G(Y_1)) & \xleftarrow{G(g)\circ - \circ f} & \mathrm{Hom}_{\mathcal{C}}(X_2, G(Y_2)) \\ {\scriptstyle \beta_{X_1,Y_1}}\downarrow & & \downarrow{\scriptstyle \beta_{X_2,Y_2}} \\ \mathrm{Hom}_{\mathcal{D}}(F(X_1), Y_1) & \xleftarrow{g\circ - \circ F(f)} & \mathrm{Hom}_{\mathcal{D}}(F(X_2), Y_2) \end{array}$$

For a given $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X_2, G(Y_2))$, we then just need to verify that

$$\psi_{Y_1}^{-1} \circ F(G(g) \circ \alpha \circ f) = g \circ \psi_{Y_2}^{-1} \circ F(\alpha) \circ F(f)$$

Since ψ is natural we have that $FG(g) \circ \psi_{Y_2} = \psi_{Y_1} \circ g$, so $\psi_{Y_1}^{-1} \circ FG(g) = g \circ \psi_{Y_2}^{-1}$. Then precomposing on both sides by $F(\alpha) \circ F(f)$ and using the functoriality of F shows that the above diagram does indeed commute and so β is a natural isomorphism as claimed.

Problem 8. Consider the functors given in the figure below. Assume that (F_1, F_2) and $(F_1, \widetilde{F_2})$ are both adjoint pairs. Prove that there is a natural isomorphism (of functors) between F_2 and $\widetilde{F_2}$.



Solution: By adjointness of (F_1, F_2) and of $(F_1, \widetilde{F_2})$ we have β and $\widetilde{\beta}$ such that for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$,

$$\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y)$$
$$\widetilde{\beta}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, \widetilde{F_2}(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y)$$

where the above maps are bijections and are natural in both X and Y. So for each $Y \in \mathcal{D}$ and $X \in \mathcal{C}$ we have a bijection

$$\mathrm{Hom}_{\mathcal{C}}(X, F_2(Y)) \xrightarrow{\beta_{X,Y}} \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y) \xrightarrow{\widetilde{\beta}^{-1}_{X,Y}} \mathrm{Hom}_{\mathcal{C}}(X, \widetilde{F_2}(Y))$$

By the naturality of the composition in X we have for each $Y \in \mathcal{D}$ a natural transformation $h_{F_2(Y)} \xrightarrow{\sim} h_{\widetilde{F_2}(Y)}$. Thinking of $h_\bullet$ as a functor from $\mathcal{C}$ to $\mathbf{Func}\,(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ then by the Yoneda lemma the natural transformation $h_{F_2(Y)} \xrightarrow{\sim} h_{\widetilde{F_2}(Y)}$ is induced by a map, $F_2(Y) \to \widetilde{F_2}(Y)$, and this map is an isomorphism (since $h_\bullet$ is full one can just pull back the inverse of the original isomorphism). Call this map $\varphi_Y : F_2(Y) \to \widetilde{F_2}(Y)$ where we aim to show that $\varphi : F_2 \to \widetilde{F_2}$ is a natural isomorphism.

It just remains to check that φ is natural in Y, i.e. that for a map $f : Y_1 \to Y_2$ the following diagram commutes

$$\begin{array}{ccc} F_2(Y_1) & \xrightarrow{F_2(f)} & F_2(Y_2) \\ {\scriptstyle \varphi_{Y_1}}\downarrow & & \downarrow{\scriptstyle \varphi_{Y_2}} \\ \widetilde{F_2}(Y_1) & \xrightarrow{\widetilde{F_2}(f)} & \widetilde{F_2}(Y_2) \end{array}$$

But by the Yoneda Lemma the category $\mathcal{C}$ is embedded into $\mathbf{Func}\,(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$, and since this diagram

$$\begin{array}{ccc} h_\bullet(F_2(Y_1)) & \xrightarrow{F_2(f)\circ -} & h_\bullet(F_2(Y_2)) \\ {\scriptstyle \widetilde{\beta}^{-1}_{-,Y_1} \circ \beta_{-,Y_1}}\downarrow & & \downarrow{\scriptstyle \widetilde{\beta}^{-1}_{-,Y_2} \circ \beta_{-,Y_2}} \\ h_\bullet(\widetilde{F_2}(Y_1)) & \xrightarrow{\widetilde{F_2}(f)\circ -} & h_\bullet(\widetilde{F_2}(Y_2)) \end{array}$$

commutes (by naturality of the β's), so does the original diagram in $\mathcal{C}$. Thus F_2 and $\widetilde{F_2}$ are naturally isomorphic as desired.

Solution: I typed up the following construction of the natural isomorphism and the check of naturality (not that it is an isomorphism) before remembering Yoneda's Lemma. I decided to leave it in because I wanted to share the pretty cartoons. Please don't grade anything from here on.

By adjointness of (F_1, F_2) and of $(F_1, \widetilde{F_2})$ we have β and $\widetilde{\beta}$ such that for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$,

$$\beta_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X, F_2(Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(F_1(X), Y)$$

$$\widetilde{\beta}_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X, \widetilde{F_2}(Y)) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(F_1(X), Y)$$

where the above maps are bijections. In particular for each $Y \in \mathcal{D}$ we have a bijection

$$\operatorname{Hom}_{\mathcal{C}}(F_2(Y), F_2(Y)) \xrightarrow{\beta_{F_2(Y),Y}} \operatorname{Hom}_{\mathcal{D}}(F_1F_2(Y), Y) \xrightarrow{\widetilde{\beta}^{-1}_{F_2(Y),Y}} \operatorname{Hom}_{\mathcal{C}}(F_2(Y), \widetilde{F_2}(Y))$$

Now we are ready to define the natural isomorphism $\varphi : F_2 \to \widetilde{F_2}$. For each $Y \in \mathcal{D}$ set φ_Y to be the image of $\mathrm{id}_{F_2(Y)}$ under the composition in the above diagram, i.e. we define

$$\varphi_Y = \widetilde{\beta}^{-1}_{F_2(Y),Y} \circ \beta_{F_2(Y),Y}(\mathrm{id}_{F_2(Y)}) : F_2(Y) \to \widetilde{F_2}(Y)$$

First we verify that φ is natural. Let $Y_1, Y_2 \in \mathcal{D}$ and $f : Y_1 \to Y_2$. We need to check that the following diagram commutes

(1)
$$\begin{array}{ccc} F_2(Y_1) & \xrightarrow{F_2(f)} & F_2(Y_2) \\ {\scriptstyle \varphi_{Y_1}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{Y_2}} \\ \widetilde{F_2}(Y_1) & \xrightarrow{\widetilde{F_2}(f)} & \widetilde{F_2}(Y_2) \end{array}$$

To verify the commutativity of the diagram 1, it is helpful to consider the following larger diagram

(2)
$$\begin{array}{ccccc} \operatorname{Hom}_{\mathcal{C}}(F_2(Y_1), \widetilde{F_2}(Y_1)) & \xrightarrow{\widetilde{F_2}(f)\circ -} & \operatorname{Hom}_{\mathcal{C}}(F_2(Y_1), \widetilde{F_2}(Y_2)) & \xleftarrow{-\circ F_2(f)} & \operatorname{Hom}_{\mathcal{C}}(F_2(Y_2), \widetilde{F_2}(Y_2)) \\ {\scriptstyle \widetilde{\beta}^{-1}_{F_2(Y_1),Y_1}}\big\uparrow & & {\scriptstyle \widetilde{\beta}^{-1}_{F_2(Y_1),Y_2}}\big\uparrow & & {\scriptstyle \widetilde{\beta}^{-1}_{F_2(Y_2),Y_2}}\big\uparrow \\ \operatorname{Hom}_{\mathcal{D}}(F_1F_2(Y_1), Y_1) & \xrightarrow{f\circ -} & \operatorname{Hom}_{\mathcal{D}}(F_1F_2(Y_1), Y_2) & \xleftarrow{-\circ f} & \operatorname{Hom}_{\mathcal{D}}(F_1F_2(Y_2), Y_2) \\ {\scriptstyle \beta_{F_2(Y_1),Y_1}}\big\uparrow & & {\scriptstyle \beta_{F_2(Y_1),Y_2}}\big\uparrow & & {\scriptstyle \beta_{F_2(Y_2),Y_2}}\big\uparrow \\ \operatorname{Hom}_{\mathcal{C}}(F_2(Y_1), F_2(Y_1)) & \xrightarrow{F_2(f)\circ -} & \operatorname{Hom}_{\mathcal{C}}(F_2(Y_1), F_2(Y_2)) & \xleftarrow{-\circ F_2(f)} & \operatorname{Hom}_{\mathcal{C}}(F_2(Y_2), F_2(Y_2)) \end{array}$$

The commutativity of every small square in 2 follows from the naturality of β or $\widetilde{\beta}$ (possibly inverted). Now observe that the lower composition in 1 is the image of $\mathrm{id}_{F_2(Y_1)}$ under the leftmost composition in 2 (from bottom left to top middle),

$$\widetilde{F_2}(f) \circ \left(\widetilde{\beta}^{-1}_{F_2(Y_1),Y_1} \circ \beta_{F_2(Y_1),Y_1}(\mathrm{id}_{F_2(Y_1)})\right).$$

Similarly the upper composition in 1 is the image of $\mathrm{id}_{F_2(Y_2)}$ under the rightmost composition in 2 (from bottom right to top middle),

$$\left(\widetilde{\beta}^{-1}_{F_2(Y_2),Y_2} \circ \beta_{F_2(Y_2),Y_2}(\mathrm{id}_{F_2(Y_2)})\right) \circ F_2(f).$$

By the commutativity of all of the squares in 2, we just need to check that the identity morphism map to the same place under the innermost composition. It now suffices to observe that $F_2(f) \circ \mathrm{id}_{F_2(Y_1)} = \mathrm{id}_{F_2(Y_2)} \circ F_2(f)$ because then the image of the identity morphisms will be the same under the initial maps left/right in 2, and thus the entire composition yields the same element. This shows that 1 does indeed commute and so φ is natural.