6112 Homework 3

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Problem 1. (4) Let $I = \{1, 2, 3, ...\}$ together with the usual order. Consider the following inverse system on I valued in **Ab**, denoted by $\mathfrak{Z} = (\{Z_n\}_{n\in I}; \{\varphi_{n,m}\}_{n\leq m})$:
$Z_n = \mathbb{Z}$ for every $n \in I$.
For every $n \leq m$, the group homomorphism $\varphi_{n,m} : Z_m \to Z_n$ is given by $\varphi_{n,m}(x) = 3^{m-n}x$.
Prove that $\varprojlim_{(I,\leq)} \mathfrak{Z} = (0)$

Proof. First it is easy to verify that this is an inverse system, since $\varphi_{n,n}(x) = 3^{n-n}x = x$ and so $\varphi_{n,n} = Id$ for all $n \in \mathbb{N}$, and next for any $n \leq m \leq \ell$ we have $\varphi_{n,\ell}(x) = 3^{\ell-n}x = 3^{(\ell-m)+(m-n)}x = \varphi_{n,m}(3^{\ell-m}x) = \varphi_{n,m} \circ \varphi_{m,\ell}(x)$ and so $\varphi_{n,\ell} = \varphi_{n,m} \circ \varphi_{m,\ell}$.

Next we will show that (0) is the inverse limit by showing that it has the universal property (for inverse limits). Define group homomorphisms $f_i : (0) \to Z_i = \mathbb{Z}$ such that $f_i(0) = 0$ (f_i is the zero homomorphism for all $i \in \mathbb{N}$). Observe that $\varphi_{i,j}f_j(0) = 0 = f_i(0)$ for all $i \leq j$ and so we see that $\varphi_{i,j}f_j = f_i$ for all $i \leq j$. Now let $(g_i)_{i\in\mathbb{N}}$ be a collection of group homomorphisms $g_i : G \to Z_i = \mathbb{N}$ (where G is some fixed abelian group independent of $i \in \mathbb{N}$)such that $\varphi_{i,j}g_j = g_i$ for all $i \leq j$, and let $g : G \to (0)$ be the zero homomorphism. Observe that for all $i \in \mathbb{N}$ that $g_i = f_i \circ g$, i.e. $g_i = 0$ (the zero homomorphism). To see this, observe that for any $x \in G$ and any $j \geq i$ that $g_i(x) = \varphi_{i,j}(g_j(x)) = 3^{j-i}g_j(x)$. Therefore, taking $j = i + n$ for any $n \in \mathbb{N}$ we see that $g_i(x) = 3^n g_{i+n}(x)$ for all $n \in \mathbb{N}$ and so $3^n \mid g_{i+n}(x)$ for all $n \in \mathbb{N}$. Therefore this implies that $g_i(x) = 0$ (if $g_i(x) \neq 0$ then it has a unique prime factorization and the prime 3 factors into it only finitely many times. Alternatively if $g_i(x) \neq 0$ then for all n $g_{i+n}(x) \in \mathbb{Z} \setminus \{0\}$ and so $|g_i(x)| = 3^n|g_{i+n}(x)| \geq 3^n$ for all $n \in \mathbb{N}$ which implies that $g_i(x)$ is an integer which is greater than all other integers, and such an integer does not exist). So since $i \in \mathbb{N}$ and $x \in G$ are arbitrary this implies that $g_i = 0$ and so we have shown that there exists $g : G \to (0)$ such that $g_i = f_i \circ g$ for all $i \in \mathbb{N}$. Moreover we see immediately that this g is the unique such group homomorphism (because the zero homomorphism is the only group homomorphism between any group G and (0)). Therefore we have found morphisms $f_i : (0) \to Z_i$ (with $\varphi_{i,j}f_j = f_i$ for all $i \leq j$) such that for any family of morphisms $(g_i)_{i\in I}$ with $g_i : G \to Z_i$ for all i and $\varphi_{i,j}g_j = g_i$ for all $i \leq j$, there exists a unique $g : G \to 0$ such that $(g_i)_{i\in I} = (f_i \circ g)_{i\in I}$. This is exactly the universal property for inverse limits, and therefore we have shown that (0) has the universal property for the inverse system $\mathfrak{Z}$, i.e. that $(0) = \varprojlim_{(I,\leq)} \mathfrak{Z}$ (the unique inverse limit up to group isomorphism). $\square$

Problem 2. (8) Prove that the inverse limit of injective morphisms is injective. Prove that the direct limit of surjective morphisms is surjective.

Proof. **Part 1:** First we define the inverse limit of morphisms. Fix $(I, \leq)$ a pre-ordered set, let $\{X_i\}_{i\in I}$, $\{\varphi_{i,j}\}_{i\leq j}$ an inverse system (with X_i objects, $\varphi_{i,j} : X_j \to X_i$ morphisms for $i \leq j$) and let $\{Y_i\}_{i\in I}$, $\{\psi_{i,j}\}_{i\leq j}$ another inverse system such that both $\{X_i\}$ and $\{Y_i\}$ have inverse limits $\varprojlim X_i$ and $\varprojlim Y_i$. For any family of morphisms $(f_i)_{i\in I}$ such that $f_i : X_i \to Y_i$ for all $i \in I$ and $f_i \circ \varphi_{i,j} = \psi_{i,j} \circ f_j$ for all $i \leq j$, we will define a morphism $\varprojlim f_i : \varprojlim X_i \to \varprojlim Y_i$ as follows. By the universal property for inverse limits we know there exist canonical maps $\alpha_j : \varprojlim X_i \to X_j$, $\beta_j : \varprojlim Y_i \to Y_j$ with $\varphi_{i,j}\alpha_j = \alpha_i$, $\psi_{i,j}\beta_j = \beta_i$ for all $i \leq j$ (and moreover the rest of the universal properties hold for $(\alpha_i)_{i\in I}$, $(\beta_i)_{i\in I}$). So for all $i \in I$ we have

$f_i \circ \alpha_i : \varprojlim X_i \to Y_i$. So we see that $(f_i \circ \alpha_i)_{i\in I}$ is a family of morphisms with $f_i \circ \alpha_i : \varprojlim X_i \to Y_i$ for all i, $\psi_{i,j} \circ f_j \circ \alpha_j = f_i \circ \varphi_{i,j} \circ \alpha_j = f_i \circ \alpha_i$ for all $i \leq j$, and therefore by the universal property for $\varprojlim Y_i$ there exists a unique $f : \varprojlim X_i \to \varprojlim Y_i$ such that $(f_i \circ \alpha_i)_{i\in I} = (\beta_i \circ f)_{i\in I}$. This unique f provided by the universal properties will be denoted $\varprojlim f_i$, and so we have defined the mapping $(f_i)_{i\in I} \mapsto \varprojlim f_i$ (where $f_i : X_i \to Y_i$ with $f_i \circ \varphi_{i,j} = \psi_{i,j} \circ f_j$).

Now that we have defined the inverse limit of morphisms we will prove the first result. Assume $f_i : X_i \to Y_i$ are injective morphisms (with $f_i \circ \varphi_{i,j} = \psi_{i,j} \circ f_j$). Now assume $g_1, g_2 : Z \to \varprojlim X_i$ such that $\varprojlim f_i \circ g_1 = \varprojlim f_i \circ g_2$. Therefore we see that

$$(f_j \circ \alpha_j \circ g_1)_{j\in I} = (\beta_j \circ \varprojlim f_i \circ g_1)_{j\in I} = (\beta_j \circ \varprojlim f_i \circ g_2)_{j\in I} = (f_j \circ \alpha_j \circ g_2)_{j\in I}$$

and so for all $j \in I$ we have $f_j \circ \alpha_j \circ g_1 = f_j \circ \alpha_j \circ g_2$. Next since each f_j is injective, by the definition of injectivity this implies $(\alpha_i \circ g_1)_{i\in I} = (\alpha_i \circ g_2)_{i\in I}$. Now observe that $\alpha_i \circ g_1 : Z \to X_i$ for all $i \in I$ with $\varphi_{i,j}\alpha_i \circ g_1 = \alpha_j \circ g_1$ for all $i \leq j$, and so by the universal property for $\varprojlim X_i$ there exists a unique $t : Z \to \varprojlim X_i$ such that $(\alpha_i \circ g_1)_{i\in I} = (\alpha_i \circ t)$. However since $(\alpha_i \circ g_1)_{i\in I} = (\alpha_i \circ g_2)_{i\in I}$ with $g_1, g_2 : Z \to \varprojlim X_i$ we see that both g_1 and g_2 satisfy this condition for t and so by uniqueness we see that $t = g_1 = g_2$. So we have shown that $\varprojlim f_i \circ g_1 = \varprojlim f_i \circ g_2$ implies $g_1 = g_2$ and thus by definition $\varprojlim f_i$ is injective. So we have shown that $(f_i)_{i\in I}$ injective (with $\psi_{i,j} \circ f_j = f_i \circ \varphi_{i,j}$ for all $i \leq j$) implies that $\varprojlim f_i$ is injective.

Part 2: This is very similar to the first part. Again we will first define the direct limit of morphisms. Let $(I, \leq)$ a directed set, $\{X_i\}_{i\in I}, \{\varphi_{j,i}\}_{i\leq j}$ and $\{Y_i\}_{i\in I}, \{\psi_{j,i}\}_{i\leq j}$ be two direct systems which admit direct limits $\varinjlim X_i$ and $\varinjlim Y_i$. Let $(f_i)_{i\in I}$ a family of morphisms $f_i : X_i \to Y_i$ such that $\psi_{j,i} \circ f_i = f_j \circ \varphi_{j,i}$ for all $i \leq j$. We define the direct limit of $(f_i)_{i\in I}$, denoted $\varinjlim f_i$ as follows. First by the universal property for direct limits we know that there exists canonical maps $\alpha_j : X_j \to \varinjlim X_i$, $\beta_j : Y_j \to \varinjlim X_i$ such that $\alpha_j \circ \varphi_{j,i} = \alpha_i$ for all $i \leq j$, $\beta_j \circ \psi_{j,i} = \beta_i$ for all $i \leq j$ and such that the rest of the universal property holds for both families α and β for $\varinjlim X_i$ and $\varinjlim Y_i$ respectively. Now we have that $\beta_j \circ f_j : X_j \to \varinjlim Y_i$ such that for all $i \leq j$,

$$\beta_j \circ f_j \circ \varphi_{j,i} = \beta_j \circ \psi_{j,i} \circ f_i = \beta_i \circ f_i$$

and so by the universal property for $\varinjlim X_i$ there exists a unique $f : \varinjlim X_i \to \varinjlim Y_i$ such that $(\beta_i \circ f_i)_{i\in I} = (f \circ \alpha_i)_{i\in I}$. Denote this f as $\varinjlim f_i$, and so by the above construction using the universal properties we have defined a map $(f_i)_{i\in I} \mapsto \varinjlim f_i$ (assuming that the family (f_i) satisfies the above conditions).

Now assume that $(f_i)_{i\in I}$ is a family of surjective morphisms $X_i \to Y_i$ with $\psi_{j,i} f_i = f_j \circ \varphi_{j,i}$ for all $i \leq j$. We want to show that $\varinjlim f_i$ is surjective also. Let $g_1, g_2 : \varinjlim Y_i \to Z$ such that $g_1 \circ \varinjlim f_i = g_2 \circ \varinjlim f_i$. So by definition of $\varinjlim f_i$ we have that:

$$(g_1 \circ \beta_j \circ f_j)_{j\in I} = (g_1 \circ \varinjlim f_i \circ \alpha_j)_{j\in I} = (g_2 \circ \varinjlim f_i \circ \alpha_j)_{j\in I} = (g_2 \circ \beta_j \circ f_j)_{j\in I}.$$

Since f_i is surjective for all $i \in I$, this implies that $(g_1 \circ \beta_i)_{i\in I} = (g_2 \circ \beta_i)_{i\in I}$. Next observe that $g_1 \circ \beta_i : Y_i \to Z$ such that for all $i \leq j$, $g_1 \circ \beta_j \psi_{j,i} = g_1 \circ \beta_i$. Therefore by the universal property for $\varinjlim Y_i$ there exists a unique $t : \varinjlim Y_I \to Z$ such that $(g_1 \circ \beta_i)_{i\in I} = (t \circ \beta_i)_{i\in I}$. However we see that both g_1, g_2 have the properties of t, and so by uniqueness of t this implies that $t = g_1 = g_2$. So we have shown that $g_1 \circ \varinjlim f_i = g_2 \circ \varinjlim f_i$ implies that $g_1 = g_2$ (where g_1, g_2 are any morphisms $\varinjlim Y_i \to Z$ for some object Z). Therefore by definition $\varinjlim f_i$ is surjective. So we have shown that $(f_i)_{i\in I}$ a family of surjective morphisms implies that $\varinjlim f_i$ is surjective (assuming the family of morphisms admits a direct limit). $\square$

Problem 3. (9) Let $\mathfrak{X}$ be a direct system over a preordered set $(I, \leq)$ valued in **Sets**. Let $X = \sqcup_{i\in I} X_i / \sim$ where the equivalence relation is:

$$x \in X_i \sim \psi_{j,i}(x) \in X_j \text{ for every } i \leq j$$

Prove that X is isomophic to $\varinjlim_{(I,\leq)} \mathfrak{X}$

Proof. First to treat this problem rigorously, define $\sqcup_{i\in I}X_i$ to be a set along with canonical inclusions $f_{i,0} : X_i \to \sqcup_{i\in I}X_i$ such that the images $f_{i,0}$ form a disjoint union $\sqcup_{i\in I}X_i = \bigcup_{i\in I} f_{i,0}(X_i)$ and such that each $f_{i,0}$ is an injection (one can show from elementary set theory that a disjoint union always exists and that it is unique up to set bijections, i.e. isomorphism in the category of sets). Next let $\sim$ be the equivalence induced by the relation given in the problem. That is, let $\sim_0$ be the relation $f_{i,0}(x) \sim_0 f_{j,0}(\psi_{j,i}(x))$ for all $x \in X_i$, $i \leq j$, and viewing the relation as a subset of the cartesian product (i.e. $\sim_0 \subset \sqcup_{i\in I}X_i \times \sqcup_{i\in I}X_i$), we define $\sim$ to be the intersection of all equivalence relations containing $\sim_0$. Observe that this is not an empty intersection, as the equivalence relation which puts all elements of $\sqcup_{i\in I}X_i$ in the same equivalence class contains $\sim_0$. Also observe that this implies that $\sim$ is the following equivalence relation. For any $x, y \in \sqcup_{i\in I}X_i$, $x \sim y$ if and only if there exists $x_0, ..., x_n \in \sqcup_{i\in I}X_i$ with $x_0 = x$, $x_n = y$ and such that for all $0 \leq k \leq n-1$ we have $x_k \sim_0 x_{k+1}$ or $x_{k+1} \sim_0 x_k$ (so in order to extend $\sim_0$ to an equivalence relation we need to force it to be symmetric and transitive by looking at all finite paths of $\sim_0$ related elements). Let $\sim'$ denote the relation defined by the right hand side (and so we will show that $\sim' = \sim$). It is clear that if $x \sim' y$, i.e. if such a path $x_0, ..., x_n$ exists, then any equivalence relation $\mathcal{R}$ extending $\sim_0$ must have $x_0\mathcal{R}x_1\mathcal{R}\cdots\mathcal{R}x_n$ and so by transitivity $x\mathcal{R}y$. Thus, $\sim' \subset \sim$, and so all that remains is to show that $\sim'$ is itself an equivalence relation (and thus is in the intersection defining $\sim$ which implies $\sim \subset \sim'$). Clearly $\sim'$ is reflexive (take $n = 1$ and look at $x_0 = x_1 = x$, and so since $\sim_0$ is reflexive we have $x_0 \sim_0 x_1$). Next observe that it is symmetric (if we have some $x = x_0, ..., x_n = y$ then just take $y_k = x_{n-k}$ for all $0 \leq k \leq n$ and so $y = y_0, ..., y_n = x$ and for all k either $y_k \sim_0 y_{k+1}$ or $y_{k+1} \sim_0 y_k$). Finally for transitivity, if we have x,y equivalent and x, z equivalent, then we have some $x = x_0, ..., x_n = y$ and $y = y_0, ..., y_m = z$ and so this gives us the sequence $x = x_0, ..., x_n, y_1, ..., y_m = z$ such that every adjacent pair of elements are related by $\sim_0$. Thus $\sim'$ is an equivalence relation, and so $\sim \subset \sim' \subset \sim$ and therefore $\sim = \sim'$. So we have descreibed the equivalence relation $\sim$ explicitly (for any elements $x, y \in \sqcup_{i\in I}X_i$ we see that $x \sim y$ if and only if there is a finite sequence of elements in $\sqcup_{i\in I}X_i$ which starts from x, ends at y, and such that every adjacent pair of elements is related by $\sim_0$). So we can define $X = \sqcup_{i\in I}X_i/\sim$ and the quotient map $Q : \sqcup_{i\in I}X_i \to X$ which sends any element x in the domain to the equivalence class $\overline{x}$ containing x.

We know that the direct limit (if it exists) is unique up to isomorphism and that an object is (isomorphic to) the direct limit if it has the universal property for the direct system. So we will show that X has the universal property for the direct system $\mathfrak{X}$. Recall that morphisms in **Sets** are set maps and isomorphisms are set bijections. Recall we gave inclusion maps $f_{i,0} : X_i \to \sqcup_{i\in I}X_i$ with the definition of the disjoint union, and the quotient map Q, and so composing these two together we have a family of maps $f_i = Q \circ f_{i,0} : X_i \to X$. Observe that for all $i \leq j$ that $f_j\psi_{j,i} = f_i$, since for any $x \in X_i$ we have by definition of $\sim_0$, $f_{i,0}(x) \sim_0 f_{j,0}(\psi_{j,i}(x))$, and thus $f_{i,0}(x) \sim f_{j,0}(\psi_{j,i}(x))$ and so by the definition of the quotient map Q we have $Q(f_{i,0}(x)) = Q(f_{j,0}(\psi_{j,i}(x)))$. Since $x \in X_i$ is arbitrary and $Q \circ f_{i,0} = f_i$, $Q \circ f_{j,0} = f_j$ we have shown that $f_i = f_j \circ \psi_{j,i}$. Therefore $f_i = f_j \circ \psi_{j,i}$ for all $i \leq j$.

Now let $(g_i)_{i\in I}$ be a family of set maps $g_i : X_i \to Y$ (where Y is some set), such that $g_j \circ \psi_{j,i} = g_i$ for all $i \leq j$. Define a set map $g_0 : \sqcup_{i\in I}X_i \to Y$ as follows. For all $i \in I$, $x_i \in X_i$ define $g_0(f_{i,0}(x_i)) = g_i(x_i)$. Observe that since $f_{i,0}$ is injective for all i and that since the images $f_{i,0}(X_i)$ form a disjoint union equal to $\sqcup_{i\in I}X_i$, this gives us a well defined map $g_0 : \sqcup_{i\in I}X_i \to Y$ (basically g_0 is a piecewise function defined on pieces indexed by I). Now observe that g_0 is constant on equivalence classes of $\sim$. To see this, first assume that $x \sim_0 y$, and so $x = f_{i,0}(x_i)$ for some $x_i \in X_i$ and $y = f_{j,0}(\psi_{j,i}(x_i))$ for some $j \geq i$. Therefore we see that

$$g_0(x) = g_0(f_{i,0}(x_i)) = g_i(x_i) = g_j(\psi_{j,i}(x_i)) = g_0(f_{j,0}(\psi_{j,i}(x_i))) = g_0(y)$$

(since $g_i = g_j \circ \psi_{j,i}$ by assumption). So we have shown that $x \sim_0 y$ implies $g_0(x) = g_0(y)$. Now assume $x \sim y$, and so there exists a finite sequence $x = x_0, ..., x_n = y$ such that for every $0 \leq k \leq n-1$, $x_k \sim_0 x_{k+1}$ or $x_{k+1} \sim_0 x_k$. Therefore by above, $g_0(x_k) = g_0(x_{k+1})$ or $g_0(x_{k+1}) = g_0(x_k)$ for all $0 \leq k \leq n$ and therefore we see that $g_0(x) = g_0(x_0) = ... = g_0(x_n) = g_0(y)$. So we have shown that $x \sim y$ implies $g_0(x) = g_0(y)$. Therefore we see that g_0 descends to the well defined set map $g : X \to Y$, $g(\overline{x}) = g_0(x)$ (where $\overline{x}$ denotes the equivalence class of $x \in \sqcup_{i\in I}X_i$, g is well defined because $x \sim y$ implies $g_0(x) = g_0(y)$). We will show that g is the unique map from X to Y such that $(g_i)_{i\in I} = (g \circ f_i)_{i\in I}$.

Fix $i \in I$ and $x_i \in X_i$, and observe that

$$g(f_i(x_i)) = g(Q(f_{i,0}(x_i))) = g_0(f_{i,0}(x_i)) = g_i(x_i).$$

So since $x_i \in X_I$ and $i \in I$ arbitrary this shows that $g \circ f_i = g_i$ for all $i \in I$. So we have shown that there exists a map $g : X \to Y$ such that $(g_i)_{i\in I} = (g \circ f_i)_{i\in I}$. Next to show uniqueness, observe that if $h \circ f_i = g_i$ for all $i \in I$, then for any $x_i \in X_i$ we have $g_i(x_i) = h(f_i(x_i))$ and $g_i(x_i) = g(f_i(x_i))$., and so $h(f_i(x_i)) = g(f_i(x_i))$ for all $i \in I$, $x_i \in X_i$. Therefore, h and g agree on the image of f_i for all $i \in I$ (i.e. $h(y) = g(y)$ for all $y \in f_i(X_i)$, $i \in I$), but observe that $X = \bigcup_{i\in i} f_i(X_i)$ (since $\sqcup_{i\in I}X_i = \bigcup_{i\in I} f_{i,0}(X_i)$ and $Q(\sqcup_{i\in I}X_i) = X$) and so this implies that $h = g$ (since these are set maps we have been able to examine them pointwise to show this uniqueness). So we have shown uniqueness.

Therefore we have chosen a family of set maps $(f_i)_{i\in I}$ such that $f_i : X_i \to X$ with $f_j\psi_{j,i} = f_i$ for all $i \leq j$, and such that for any family $(g_i)_{i\in I}$ of set maps $g_i : X_i \to Y$ with Y an arbitrary set and $g_j \circ \psi_{j,i} = g_i$ for all $i \leq j$, there exists a unique $g : X \to Y$ such that $(g_i)_{i\in I} = (g \circ f_i)_{i\in I}$. So we have shown exactly the universal property for direct limits, and thus we conclude that $X = \varinjlim \mathfrak{X}$ (the unique direct limit up to isomorphism). □

Problem 4. (10) With the set up of Problem 9 above, assume that I is right directed and each X_i has a structure of a group and each $\psi_{j,i}$ is a group homomorphism. Prove that X has a natural structure of a group which makes it isomorphic to the direct limit of $\mathfrak{X}$ in the category of groups.

Proof. Define the disjoint union $\sqcup_{i\in I}X_i$ with canonical inclusion maps $f_{i,0} : X_i \to \sqcup_{i\in I}X_i$ in the same way as in the previous problem, and again let $\sim$ be the equivalence relation induced by $\sim_0$ in the same way as the previous problem, with $Q : \sqcup_{i\in I}X_i \to X$ the quotient map and $f_i = Q \circ f_{i,0} : X_i \to X$ for all $i \in I$. Observe that with the additional assumption that I is right directed, we get that $\sim$ is an even simpler equivalence relation as follows: $x \sim y$ if and only if there exists $x_i \in X_i$ and $y_j \in X_j$ such that $x = f_{i,0}(x_i)$, $y = f_{j,0}(y_j)$, and there exists $k \geq i,j$ such that $\psi_{k,i}(x_i) = \psi_{k,j}(x_j)$. Clearly we see that $f_{i,0}(x_i) \sim_0 f_{k,0}(\psi_{k,i}(x_i))$ and $f_{j,0}(x_j) \sim_0 f_{k,0}(\psi_{k,j}(x_j))$ and so for any equivalence relation $\mathcal{R}$ extending $\sim_0$ we have $f_{j,0}(x_j)\mathcal{R}f_{j,0}(x_j)$, i.e. $x\mathcal{R}y$. Additionally observe that the properties defined above give an equivalence relation $\sim'$ (i.e. $x \sim' y$ if there exists $x_i \in X_i$ and $y_j \in X_j$ such that $x = f_{i,0}(x_i)$, $y = f_{j,0}(y_j)$, and there exists $k \geq i,j$ such that $\psi_{k,i}(x_i) = \psi_{k,j}(x_j)$). To see this note that $\sim'$ is clearly reflexive and symmmetric, and for transitivity if $f_{i,0}(x_i) \sim' f_{j,0}(x_j) \sim' fk,0(x_k)$, then there exists $m \geq i,j$ and $n \geq j,k$ such that $\psi_{m,i}(x_i) = \psi_{m,j}(x_j)$ and $\psi_{n,j}(x_j) = \psi_{n,k}(x_k)$, and moreover there exists $\ell \geq n,m$ by I right directed, and therefore

$$\psi_{\ell,i}(x_i) = \psi_{\ell,m}\psi_{m,i}(x_i) = \psi_{\ell,m}\psi_{m,j}(x_j) = \psi_{\ell,j}(x_j) = \psi_{\ell,n}\psi_{n,j}(x_j) = \psi_{\ell,n}\psi_{n,k}(x_k) = \psi_{\ell,k}(x_k)$$

and so $f_{i,0}(x_i) \sim' f_{k,0}(x_k)$. So we have that $\sim'$ is an equivalence relation contained by all equivalence relations $\mathcal{R}$ containing $\sim_0$, and therefore we have $\sim' = \sim$. So with the assumption that I is right directed we get a nicer induced relation $\sim$ on $\sqcup_{i\in I}X_i$ and we will use this to define a group structure on X.

Define multiplication on X as follows. For any $i,j \in I$ and $x_i \in X_i$, $x_j \in X_j$ define $f_i(x_i) * f_j(x_j) = f_k(\psi_{k,i}(x_i)\psi_{k,j}(x_j))$ where $k \in I$ such that $k \geq i,j$. Since I is right directed there exists such a k, since each X_i is a group the multiplication $\psi_{k,i}(x_i)\psi_{k,j}(x_j)$ makes sense as multiplication in X_k, but we need to show that the multiplication on X is well defined (i.e. the result is independent of what k you use and is also independent of what entries in the equivalence class of $f_i(x_i)$ and $f_j(x_j)$ you use). Assume that $k,\ell \in I$ with $k \geq i,j$ and $\ell \geq i,j$ and examine the two elements $f_k(\psi_{k,i}(x_i)\psi_{k,j}(x_j))$, $f_\ell(\psi_{\ell,i}(x_i)\psi_{\ell,j}(x_j))$. Observe that there exists $m \geq k,\ell$ and so we have $f_m\psi_{m,k} = f_k$ and $f_m\psi_{m,\ell} = f_\ell$ (proved last problem) and note that by definition of the $(f_i)_{i\in I}$ we know that $f_k(\psi_{k,i}(x_i)\psi_{k,j}(x_j))$ is the equivalency class containing $f_{k,0}(\psi_{k,i}(x_i)\psi_{k,j}(x_j))$ and $f_\ell(\psi_{\ell,i}(x_i)\psi_{\ell,j}(x_j))$ is the equivalency class containing $f_{\ell,0}(\psi_{\ell,i}(x_i)\psi_{\ell,j}(x_j))$.

Observe that

$$\psi_{m,k}(\psi_{k,i}(x_i)\psi_{k,j}(x_j)) = \psi_{m,k}(\psi_{k,i}(x_i))\psi_{m,k}(\psi_{k,j}(x_j)) = \psi_{m,i}(x_i)\psi_{m,j}(x_j)$$

(using the extra assumption that the family ψ are group homomorphisms) and

$$\psi_{m,\ell}(\psi_{\ell,i}(x_i)\psi_{\ell,j}(x_j)) = \psi_{m,\ell}(\psi_{\ell,i}(x_i))\psi_{m,\ell}(\psi_{\ell,j}(x_j)) = \psi_{m,i}(x_i)\psi_{m,j}(x_j).$$

Therefore $\psi_{m,k}(\psi_{k,i}(x_i)\psi_{k,j}(x_j)) = \psi_{m,\ell}(\psi_{\ell,i}(x_i)\psi_{\ell,j}(x_j))$ and so by the definition of $\sim$ which we showed above, this implies that $f_{k,0}(\psi_{k,i}(x_i)\psi_{k,j}(x_j)) \sim f_{\ell,0}(\psi_{\ell,i}(x_i)\psi_{\ell,j}(x_j))$ i.e. that $f_k(\psi_{k,i}(x_i)\psi_{k,j}(x_j)) = f_\ell(\psi_{\ell,i}(x_i)\psi_{\ell,j}(x_j))$. Next we want to show that it is independent of the choice of entry in the equivalency class (i.e. that we have defined multiplication between equivalency classes rather than between fixed representatives of the equivalency classes). Assume $f_{i,0}(x_i) \sim f_{j,0}(x_j)$ and $f_{k,0}(x_k) \sim f_{\ell,0}(x_\ell)$, and we want to show that $f_i(x_i) * f_k(x_k) = f_j(x_j) * f_\ell(x_\ell)$. That is, we want to show that $f_{m,0}(\psi_{m,i}(x_i)\psi_{m,k}(x_k)) \sim f_{n,0}(\psi_{n,j}(x_j)\psi_{n,\ell}(x_\ell))$ where $n \geq j, \ell$ and $m \geq i, k$ (using the previous fact that our choice of m and n doesn't matter). To see this, again by I right directed we can choose some $t \geq m, n, i, j, k, \ell$ such that $\psi_{t,i}(x_i) = \psi_{t,j}(x_j)$ and $\psi_{t,k}(x_k) = \psi_{t,\ell}(x_\ell)$. Thus we see that:

$$\psi_{t,m}(\psi_{m,i}(x_i)\psi_{m,k}(x_k)) = \psi_{t,i}(x_i)\psi_{t,k}(x_k) = \psi_{t,j}(x_j)\psi_{t,\ell}(x_\ell) = \psi_{t,n}(\psi_{n,j}(x_j)\psi_{n,\ell}(x_\ell)).$$

and so by the definition of $\sim$ and $*$ we have shown $f_i(x_i) * f_k(x_k) = f_j(x_j) * f_\ell(x_\ell)$. So we have a well defined binary operator $*\colon X \times X \to X$. Now we want to show that $*$ has the required properties for group multiplication (associativity, identity, inverses).

Associativity: Let $x, y, z \in X$ and so we want to show that $(x * y) * z = x * (y * z)$. There exists $i, j, k \in I$, $x_i \in X_i$, $x_j \in X_j$, $x_k \in X_k$ such that $x = f_i(x_i)$, $y = f_k(x_k)$, $z = f_j(x_j)$. Now since I is right directed we can choose $t \geq i, j, k$ (choose $m \geq i, j$, $n \geq j, k$ and $t \geq m, n$). Therefore by definition $(x{*}y) = f_t(\psi_{t,i}(x_i)\psi_{t,j}(x_j))$, and so since $t \geq t, k$ we have

$$(x * y) * z = f_t(\psi_{t,t}(\psi_{t,i}(x_i)\psi_{t,j}(x_j))\psi_{t,k}(x_k)) = f_t(\psi_{t,i}(x_i)\psi_{t,j}(x_j)\psi_{t,k}(x_k))$$

by associativity of multiplication in the group X_t (and by $\psi_{t,t}$ the identity). Similarly $y{*}z = f_t(\psi_{t,j}(x_j)\psi_{t,k}(x_k))$ and

$$x * (y * z) = f_t(\psi_{t,i}(x_i)\psi_{t,t}(\psi_{t,j}(x_j)\psi_{t,k}(x_k))) = f_t(\psi_{t,i}(x_i)\psi_{t,j}(x_j)\psi_{t,k}(x_k)).$$

Therefore $(x * y) * z = x * (y * z)$ for any $x, y, z \in X$ and so $*$ is associative.

Identity: Let $e = f_i(e_i)$ for some $i \in I$, $e_i \in X_i$ the identity element. Assume for now that i is fixed, and we will show that e is the multiplicative identity (moreover since i is arbitrary and multiplicative identities are unique this implies that $f_i(e_i) = f_j(e_j)$ for all $i, j \in I$, although we could also prove this fact directly). Let $x \in X$ arbitrary, so there exists $j \in I$ such that $x = f_j(x_j)$ for some $x_j \in X_j$. So choosing $t \geq i, j$ we have $e * x = f_t(\psi_{t,i}(e_i)\psi_{t,j}(x_j))$. However, since $\psi_{t,i}$ is a group homomorphism we have $\psi_{t,i}(e_i) = e_t$ and so $e * x = f_t(\psi_{t,j}(x_j))$. Observe that $f_{j,0}(x_j) \sim f_{t,0}(\psi_{t,j}(x_j))$ since $t \geq t, j$ with $\psi_{t,j}(x_j) = \psi_{t,t}(\psi_{t,j}(x_j)) = \psi_{t,j}(x_j)$. Therefore $f_t(\psi_{t,j}(x_j)) = f_j(x_j)$ and so we have shown that $e * x = x$. Similarly,

$$x * e = f_t(\psi_{t,j}(x_j)\psi_{t,i}(e_i)) = f_t(\psi_{t,j}(x_j)) = x.$$

So we have that $e \in X$ is an identity element (and thus it follows that it is the unique identity element).

Inverses: Fix $x \in X$, and so there exists $i \in I$ such that $x = f_i(x_i)$. Let $y = f_i(x_i^{-1})$ (where x_i^{-1} is the inverse in X_i). So, since $i \geq i, i$ we see by definition of $*$ that:

$$x * y = f_i(\psi_{i,i}(x_i)\psi_{i,i}(x_i^{-1}) = f_i(x_i x_i^{-1}) = f_i(e_i) = e$$

(recall that $e = f_i(e_i)$ for all $i \in I$) and

$$y * x = f_i(\psi_{i,i}(x_i^{-1})\psi_{i,i}(x_i)) = f_i(x_i^{-1}x_i) = f_i(e_i) = e.$$

So y acts as the inverse for x, and so we have shown that any element of X has an inverse x^{-1}.

So we have shown that X is a group, so now we want to show X has the universal property (as a direct limit of groups). We already gave set maps $f_i : X_i \to X$ such that $f_j\psi_{j,i} = f_i$ for all $i \leq j$ (by the previous problem), next we will show that they are group homomorphisms. Observe that for any $i \in I$ and $x_i, y_i \in X_i$ we have that

$$f_i(x_i) * f_i(y_i) = f_i(\psi_{i,i}(x_i)\psi_{i,i}(y_i)) = f_i(x_i y_i)$$

and so we see that $f_i : X_i \to X$ is a group homomorphism for all $i \in I$. Next assume we have a family of group homomorphisms $(g_i)_{i\in I}$ with $g_i : X_i \to H$ for all i and $g_j \circ \psi_{j,i} = g_i$ for all $i \leq j$. Therefore by the previous problem, since the $\{X_i\}$ and $\{\psi_{i,j}\}$ form a direct system as sets with direct limit X, by the universal property there is a unique set map $g : X \to H$ such that $(g_i)_{i\in I} = (g \circ f_i)_{i\in I}$. Next we want to show that g is a group homomorphism. First recall how we defined g, and next recall that for any $x, y \in X$ there exists some $x_i \in X_i$, $x_j \in X_j$ such that $x = f_i(x_i)$ and $y = f_j(x_j)$. Choose $t \geq i, j$ and so by definition of $\sim$ we have that $x = f_t(\psi_{t,i}(x_i))$ and $y = f_t(\psi_{t,j}(x_j))$, and thus $g(x) = g_t(\psi_{t,i}(x_i))$ and $g(y) = g_t(\psi_{t,j}(x_j))$. So since g_t is a group homomorphism,

$$g(x)g(y) = g_t(\psi_{t,i}(x_i))g_t(\psi_{t,j}(x_j)) = g_t(\psi_{t,i}(x_i)\psi_{t,j}(x_j)).$$

Observe that $x * y = f_t(\psi_{t,i}(x_i)\psi_{t,j}(x_j))$ and so $g(x * y) = g_t(\psi_{t,i}(x_i)\psi_{t,j}(x_j)) = g(x)g(y)$. Since $x, y \in X$ arbitrary we have shown that g is a group homomorphism. So we have shown existence of such a group homomorphism g, and uniqueness follows from the uniqueness of g as a set map (if there exists another group homomorphism $h : X \to H$ such that $(g_i)_{i\in I} = (h \circ f_i)_{i\in I}$, then by uniqueness from universal property as a direct limit of sets this implies that $h = g$ as set maps, and two group homomorphisms being equal as set maps implies they are equal as group homomorphisms).

Therefore we have shown that X is a group and it satisfies the universal property for the direct system of groups $\{X_i\}_{i\in I}$, $\{\psi_{j,i}\}_{i\leq j}$, and thus this implies that $X = \varinjlim \mathfrak{X}$ (i.e. X is the direct limit of $\mathfrak{X}$ which is unique up to group isomorphism). $\square$