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20/20 Shoot yeah!

Algebra II HW 5.

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$$\begin{tikzcd} M_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & M_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & M_3 \arrow[r, "a_3"] \arrow[d, "f_3"'] & M_4 \arrow[r, "a_4"] \arrow[d, "f_4"'] & M_5 \arrow[d, "f_5"'] \\ N_1 \arrow[r, "b_1"'] & N_2 \arrow[r, "b_2"'] & N_3 \arrow[r, "b_3"'] & N_4 \arrow[r, "b_4"'] & N_5 \end{tikzcd}$$

a) To prove if f_2, f_4 injective and f_1 is surjective $\Rightarrow$ f_3 is injective

Pf Let $x_3 \in M_3$ s.t $f_3(x_3) = 0$.

$\Rightarrow b_3 f_3(x_3) = 0 \Rightarrow f_4 a_3(x_3) = 0$ [III square commutes].

As f_4 is injective

$a_3(x_3) = 0 \Rightarrow x_3 \in \ker(a_3)$

$\Rightarrow x_3 \in \operatorname{img}(a_2)$ [$\because$ Rows are exact].

$\Rightarrow \exists\ x_2 \in M_2$ s.t. $a_2(x_2) = x_3$

$\Rightarrow f_3 a_2(x_2) = f_3(x_3) = 0$

$\Rightarrow b_2 f_2(x_2) = 0$ [II square commutes].

$\Rightarrow f_2(x_2) \in \ker(b_2) = \operatorname{Img}(b_1)$ [Rows are exact]

$\Rightarrow \exists\ y_1 \in N_1$ s.t $b_1(y_1) = f_2(x_2)$

But f_1 is surjective, i.e $\exists\ x_1 \in M_1$ s.t.

$f_1(x_1) = y_1$

$\Rightarrow b_1 f_1(x_1) = b_1(y_1) = f_2(x_2)$.

$\Rightarrow f_2 a_1(x_1) = f_2(x_2)$

$\Rightarrow a_1(x_1) = x_2$ [$\because$ f_2 is injective].

$\Rightarrow x_3 = a_2(x_2) = a_2 a_1(x_1) = 0$

[As Rows are exact]

b)

$$\begin{tikzcd}
M_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & M_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & M_3 \arrow[r, "a_3"] \arrow[d, "f_3"'] & M_4 \arrow[r, "a_4"] \arrow[d, "f_4"'] & M_5 \arrow[d, "f_5"'] \\
N_1 \arrow[r, "b_1"'] & N_2 \arrow[r, "b_2"'] & N_3 \arrow[r, "b_3"'] & N_4 \arrow[r, "b_4"'] & N_5
\end{tikzcd}$$

If f_2, f_4 are surjective, f_5 is injective
$\Rightarrow f_3$ is surjective

<u>pf</u> Let $y_3 \in N_3$. To show $\exists\ x \in M_3$ s.t $f_3(x) = y_3$

$\Rightarrow b_3(y_3) \in N_4$. As f_4 is surjective,

$\exists\ x_4 \in M_4$ s.t $f_4(x_4) = b_3(y_3)$ — (i)

$b_4 f_4(x_4) = b_4 b_3(y_3) = 0$

$\Rightarrow f_5(a_4(x_4)) = 0$

$\Rightarrow a_4(x_4) = 0$ [As f_5 is injective].

i.e $x_4 \in \ker(a_4) = \operatorname{img}(a_3)$

$\Rightarrow \exists\ x_3 \in M_3$ s.t $a_3(x_3) = x_4$

$$\begin{aligned}
b_3\big(y_3 - f_3(x_3)\big) &= b_3(y_3) - b_3 f_3(x_3) \\
&= b_3(y_3) - f_4 a_3(x_3) \\
&= b_3(y_3) - f_4(x_4) \\
&= 0 \qquad \text{[From eqn (i)]}.
\end{aligned}$$

$\Rightarrow y_3 - f_3(x_3) \in \ker(b_3) = \operatorname{img} b_2$

i.e $\exists\ y_2 \in N_2$ s.t $b_2(y_2) = y_3 - f_3(x_3)$.

As f_2 is surjective, $\exists\ x_2 \in M_2$ s.t $f_2(x_2) = y_2$

$b_2 f_2(x_2) = b_2(y_2) = y_3 - f_3(x_3)$.

$\Rightarrow f_3 a_2(x_2) = y_3 - f_3(x_3)$.

$\Rightarrow y_3 = f_3(x_3 + a_2(x_2))$

Hence, f_3 is surjective.

5 C^*, D^* cochain complex of R-modules

$\alpha^* : C^* \to D^*$ morphism

$C[1]^* \in K^*(A)$ $\quad C[1]^n = C^{n+1}$ $\quad d^n_{C[1]} = -d_C^{n+1}$

$\mathrm{Cone}(\alpha)^n = C^{n+1} \oplus D^n$

$$D^n(x,y) = (-d_C^{n+1}(x), d_D^n(y) - \alpha^{n+1}(x))$$

(1) Cone(α) is a complex.

$$D^{n+1} \circ D^n(x,y) = D^{n+1}(-d_C^{n+1}(x), d_D^n(y) - \alpha^{n+1}(x)).$$
$$= (d_C^{n+2} d_C^{n+1}(x), d_D^{n+1}(d_D^n(y) - \alpha^{n+1}(x)) + \alpha^{n+2} d_C^{n+1}(x)).$$
$$= (0, 0 - (d_D^{n+1}\alpha^{n+1} - \alpha^{n+2} \cdot d_C^{n+1})(x))$$
$$= (0,0) \quad [\text{As } \alpha \text{ is a morphism}].$$

(2) Define $\Pi^* : D^* \to \mathrm{Cone}(\alpha)$.

$\forall n \in \mathbb{N}$ $\quad \Pi^n : D^n \to \mathrm{Con}(\alpha)^n = C^{n+1} \oplus D^n$

$$y \mapsto (0,d).$$

and $p^* : \mathrm{Cone}(\alpha) \to C^*[1]$

$$p^n : \mathrm{Con}(\alpha)^n \longmapsto C^n[1] = C^{n+1}$$
$$(x,y) \longmapsto x$$

For each $n \in \mathbb{N}$.

$$\begin{tikzcd}
0 \arrow[r] & D^n \arrow[r, "\Pi^n"] \arrow[d, "d_D^n"'] & \mathrm{Cone}(\alpha)^n \arrow[r, "p^n"] \arrow[d, "D^n"] & C^n[1] \arrow[r] \arrow[d, "d^n_{C[1]}"] & 0 \\
0 \arrow[r] & D^{n+1} \arrow[r, "\Pi^{n+1}"] & \mathrm{Cone}(\alpha)^{n+1} \arrow[r, "p^{n+1}"] & C^{n+1}[1] \arrow[r] & 0
\end{tikzcd}$$

The rows are clearly exact.

For $y \in D^n$

$$D^n \cdot \Pi^n(y) = D^n(0,y) = (0, d_D^n(y)).$$

and $\quad \Pi^{n+1} d_D^n(y) = (0, d_D^n(y))$

Similarly for $(x,y) \in \mathrm{Cone}(\alpha)^n$

$$d^n_{C[1]} p^n(x,y) = -d_C^{n+1}(x)$$

$$p^{n+1} D^n(x,y) = p^{n+1}(-d_C^{n+1}(x), \ldots) = -d_C^{n+1}(x).$$

Hence, the square commutes and we have ses

$$0 \to D^* \xrightarrow{\pi^*} \mathrm{Cone}(\alpha) \xrightarrow{p^*} C^*[1] \to 0$$

(3) To prove α^* is null homotopic iff the sequence above is split.

Pf $\Leftarrow$ Assume the above sequence splits

i.e $\exists\ \beta^* : C^*[1] \longrightarrow \mathrm{cone}(\alpha)$

s.t $p^* \circ \beta^* = Id_{C^*[1]}$

i.e For each $n \in \mathbb{N}$.

$$\begin{tikzcd} \mathrm{Cone}(\alpha)^n \arrow[d, "D^{n}"'] & C^n[1] \arrow[l, "\beta^n"'] \arrow[d, "d^n_{C[1]}"] \\ \mathrm{Cone}(\alpha)^{n+1} & C^{n+1}[1] \arrow[l, "\beta^{n+1}"'] \end{tikzcd}$$

commutes.

i.e $D^{n} \circ \beta^n = \beta^{n+1} d^n_{C[1]}$

Let $\beta^n = (\beta_1^n, \beta_2^n)$
where $\beta_1^n = p_1 \circ \beta^n$
$\beta_2^n = p_2 \circ \beta^n$
p_1, p_2 projection maps.

Now $D^{n} \circ \beta^n = D^{n}(\beta_1^n, \beta_2^n)$.
$$= (-d_C^{n+1} \circ \beta_1^n,\ d_D^n \circ \beta_2^n - \alpha^{n+1} \circ \beta_1^n)$$

and $\beta^{n+1} \cdot d^n_{C[1]} = (\beta_1^{n+1} \cdot (-d_C^{n+1}),\ \beta_2^{n+1} \cdot (-d_C^{n+1}))$

But $\beta_1^n = p_1 \circ \beta^n = Id_{C^n[1]}$.

Hence $d_D^n \circ \beta_2^n - \alpha^{n+1} \cancel{\beta_1^n} = - \beta_2^{n+1} \cdot d_C^{n+1}$

$\Rightarrow \alpha^{n+1} = d_D^n \cdot \beta_2^n + \beta_2^{n+1} \cdot d_C^{n+1}$

Let $s^{n+1} := \beta_2^n \quad \forall\, n \in \mathbb{N}$.

Hence α^* is null homotopic.

$\Rightarrow$ Let α^* be null homotopic

i.e $\exists\, s^n : C^n \longrightarrow D^{n-1} \quad \forall\, n \in \mathbb{N}$.

s.t $\alpha^n = d_D^{n-1} s^n + s^{n+1} d_C^n$

To show the sequence split we have to define $\beta^* : C^*[1] \longrightarrow Cone(\alpha)^*$

s.t $p^* \cdot \beta^* = Id_{C^n[1]}$.

For each $n \in \mathbb{N}$ define

$\beta^n = (\beta_1^n, \beta_2^n)$ where $\beta_1^n : C^n[1] \longrightarrow C^{n}[1]$.

$\beta_2^n : C^n[1] \longrightarrow D^{n}$

$p^n \cdot \beta^n = Id_{C^n[1]}$.

$\Rightarrow p^n \cdot (\beta_1^n, \beta_2^n) = \beta_1^n = Id_{C^n[1]}$.

Define $\beta_2^n := s^{n+1} : C^{n+1} \longrightarrow D^n$

Claim: β is a co-chain morphism

To show the following diagram commutes.

$$\begin{tikzcd}
Cone(\alpha)^n \arrow[d, "D^n"'] & C^n[1] \arrow[l, "\beta^n"'] \arrow[d, "d^n_{C[1]}"] \\
Cone(\alpha)^{n+1} & C^{n+1}[1] \arrow[l, "\beta^{n+1}"']
\end{tikzcd}$$

$D^n \cdot \beta^n = D^n \cdot (\beta_1^n, \beta_2^n) = D^n(Id_{C^n[1]}, s^{n+1})$

$= (-d_C^{n+1} \cdot Id_{C^n[1]}, -d_D^n \cdot s^{n+1} - \alpha^{n+1} \cdot Id_{C^n[1]})$

$$= (-d_C^{n+1},\ d_D^n \cdot s^{n+1} - \alpha^{n+1})$$
$$= (-d_C^{n+1},\ -s^{n+2} \cdot d_C^{n+1})$$

$$\beta^{n+1} \cdot d^n_{C[1]} = (\beta_1^{n+1}, \beta_2^{n+1}) \cdot (-d_C^{n+1})$$
$$= (id_{C[1]}, s^{n+2}) \cdot (-d_C^{n+1})$$
$$= (-d_C^{n+1}, -s^{n+2} \cdot d_C^{n+1})$$

$\Rightarrow \beta^{n+1} \cdot d^n_{C[1]} = D^n \cdot \beta^n$. Hence, β is a morphism.

(d) $H^n(\alpha)$ is a isomorphism $\forall n \in \mathbb{Z}$
iff $H^n(\mathrm{Cone}(\alpha)) = 0 \quad \forall n \in \mathbb{Z}$.

Pf By part (2) we have the following short exact sequence

$$0 \longrightarrow D^* \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow C^*[1] \longrightarrow 0$$

This gives a long exact sequence

$$\begin{tikzcd}
\cdots H_n(D) \arrow[r] & H_n(\mathrm{Cone}(\alpha)) \arrow[r] & H_n(C^*[1]) \arrow[dll, "\delta"'] \\
H_{n+1}(D) \arrow[r] & H_{n+1}(\mathrm{Cone}(\alpha)) \arrow[r] & H_{n+1}(C[1]) \cdots
\end{tikzcd}$$

Claim: $\delta = -H_n(\alpha)$.

$$\begin{tikzcd}
 & & & \mathrm{Ker}(d^n_{C[1]}) \arrow[d, "i"] & \\
0 \arrow[r] & D^n \arrow[r, "\pi^n"] \arrow[d, "d_D^n"] & \mathrm{Cone}(\alpha)^n \arrow[r, "p^n"] \arrow[d, "B^n"] & C^n[1] \arrow[r] \arrow[d, "d^n_{C[1]}"] & 0 \\
0 \arrow[r] & D^{n+1} \arrow[r, "\pi^{n+1}"] \arrow[d, "\eta"] & \mathrm{Cone}(\alpha)^{n+1} \arrow[r, "p^{n+1}"] & C^{n+1}[1] \arrow[r] & 0 \\
 & D^{n+1}/\mathrm{Img}(d_D^n) & & &
\end{tikzcd}$$

Let $x \in \mathrm{Ker}(d^n_{C[1]})$

By the proof of snake lemma.

$$\delta' : \mathrm{Ker}(d^n_{C[1]}) \longrightarrow D^{n+1}/\mathrm{Img}(d^n_D)$$

$$x \longmapsto \eta \cdot (\pi^{n+1})^{-1} D^n \cdot (p^n)^{-1} \cdot i(x).$$

is well defined.

$$\Rightarrow \delta'(x) = \eta \cdot (\pi^{n+1})^{-1} D^n (p^n)^{-1} x$$

$$= \eta \cdot (\pi^{n+1})^{-1} D^n (x, 0) \qquad [\text{choice of } (x,b) \text{ does not matter}].$$

$$= \eta \cdot (\pi^{n+1})^{-1} \left[-d^{n+1}_c x,\ 0 - \alpha^{n+1}(x) \right].$$

$$= \eta \cdot (\pi^{n+1})^{-1} [0, -\alpha^{n+1}(x)] \qquad [\text{As } x \in \mathrm{Ker}\, d^n_{C[1]}]$$

$$= \eta \cdot -\alpha^{n+1}(x)$$

$$= [-\alpha^{n+1}(x)]$$

Also $d^{n+1}_D(-\alpha^{n+1}(x)) = -\alpha^{n+2} d^{n+1}_c x = 0$

$\Rightarrow -\alpha^{n+1}(x) \in \mathrm{Ker}(d^{n+1}_D)$.

$\Rightarrow [-\alpha^{n+1}(x)] \in \dfrac{\mathrm{Ker}(d^{n+1}_D)}{\mathrm{Img}\, d^n_D} \cong H^{n+1}(D)$.

Hence $\delta' : \mathrm{Ker}(d^n_{C[1]}) \longrightarrow H^{n+1}(D)$ is well defined and ~~and~~ also $\delta'(-d^n_c(y)) = [+\alpha^{n+1} d^n_c(y)]$

$$= [+d^n_D \alpha^{n+1}(y)] = 0$$

Hence, $\delta : H^n(C_{[1]}) \longrightarrow H^{n+1}(D)$

$$[x] \longmapsto [-\alpha^{n+1} x] = -H_n(\alpha) \cdot [x].$$

$\Rightarrow \delta = -H_n(\alpha)$ is the connecting morphism.

$$\Rightarrow \cdots H_n(\mathrm{Cone}(\alpha)) \xrightarrow{H_n(p)} H_n(C^*[1]) \xrightarrow{-H_n(\alpha)} H_{n+1}(D) \xrightarrow{H_{n+1}(\pi)} H_{n+1}(\mathrm{Cone}(\alpha)) \cdots$$

is an ~~short~~ exact sequence.

Hence, if $-H_n(\alpha)$ is an isomorphism ; $\mathrm{Ker}\, H_{n+1}(\pi) = \mathrm{img}\, H_n(\alpha) = H_{n+1}(D)$

$\Rightarrow \mathrm{img}\, H_{n+1}(\pi) = \{0\}$

$= \operatorname{Ker} H_{n+1}(p) = \{0\}$.

However as $-H_{n+1}(\alpha)$ is injective

$\operatorname{Img} H_{n+1}(p) = \operatorname{Ker}(-H_{n+1}(\alpha)) = \{0\}$.

$\Rightarrow \operatorname{Ker} H_{n+1}(p) = H_{n+1}(\operatorname{cone}(\alpha))$

$\Rightarrow H_{n+1}(\operatorname{cone}(\alpha)) = 0$.

Conversely if $H_n(\operatorname{cone}(\alpha)) = 0 \ \forall n \in \mathbb{N}$:

we have

$$0 \longrightarrow H_n(C[1]) \xrightarrow{-H_n(\alpha)} H_{n+1}(D) \longrightarrow 0$$

$\Rightarrow -H_n(\alpha)$ and hence $H_n(\alpha)$ is an isomorphism