MATH 6112 ALGEBRA II
PROBLEM SET 6

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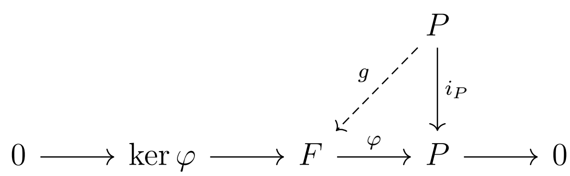
Feb 24, 2018

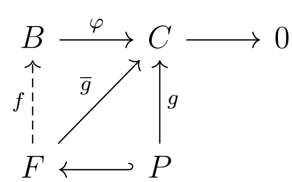
Problem 1. Let $P \in R-\mathbf{mod}$. Prove that P is projective if, and only if, there exists $P' \in R-\mathbf{mod}$ such that $P \oplus P'$ is a free R–module.

Proof. ($\Rightarrow$) $P \in R$-$\mathbf{mod}$, so there is a free module $F = \bigoplus_{a \in P} R$ and a surjection $\varphi : F \to P$. Suppose P is projective. Completing



one obtains that the horizontal sequence splits, and P is a direct summand of free module F, namely $F = P \oplus \ker\varphi$.

($\Leftarrow$) Suppose $F = P \oplus P'$ is a free module and $\varphi : B \to C$ is surjective. Given a homomorphism $g : P \to C$, we can extend g to $\overline{g} : F \to C$ by letting $\overline{g}|_{P'} = 0$. F is projective, so there is $f : P \to B$ such that $\overline{g} = \varphi f$. Then we have $g = \varphi f|_P$, which implies that P is projective.



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Problem 4. Consider the following two short exact sequences of R–modules, where P_1 and P_2 are projective.

$$0 \longrightarrow N_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow N_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0$$

Prove that $P_1 \oplus N_2$ is isomorphic to $P_2 \oplus N_1$.

Proof. Consider the diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_1 & \xrightarrow{i_1} & P_1 & \xrightarrow{\pi_1} & M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\| & & \\
0 & \longrightarrow & N_2 & \xrightarrow{i_2} & P_2 & \xrightarrow{\pi_2} & M & \longrightarrow & 0
\end{array}$$

Since P_1 is projective, there is $\beta : P_1 \to P_2$ with $\pi_2\beta = \pi_1$. Then $\pi_2\beta i_1 = \pi_1 i_1 = 0$. (N_2, i_2) is a kernel of π_2, so there is $\alpha : N_1 \to N_2$ with $\beta i_1 = i_2\alpha$. This commutative diagram with exact rows gives us a sequence

$$0 \longrightarrow N_1 \xrightarrow{f} P_1 \oplus N_2 \xrightarrow{g} P_2 \longrightarrow 0$$

where $f(n_1) = (i_1(n_1), \alpha(n_1))$ and $g(p_1, n_2) = \beta(p_1) - i_2(n_2)$.

Claim that this sequence is exact. Easily, f is injective since i_1 is injective. $gf(n_1) = \beta i_1(n_1) - i_2\alpha(n_1) = 0$, so $\operatorname{im} f \subset \ker g$. If $\beta(p_1) = i_2(n_2)$, then $\pi_1(p_1) = \pi_2\beta(p_1) = \pi_2 i_2(n_2) = 0$, so $p_1 = i_1(n_1)$ for some $n_1 \in N_1$. We have $i_2\alpha(n_1) = \beta i_1(n_1) = \beta(p_1) = i_2(n_2)$. i_2 is injective, so $\alpha(n_1) = n_2$, implying that $\ker g \subset \operatorname{im} f$. Finally, suppose $p_2 \in P_2$. Then $\pi_2(p_2) = \pi_1(p_1)$ for some $p_1 \in P_1$ and $\pi_2(\beta(p_1) - p_2) = \pi_1(p_1) - \pi_2(p_2) = 0$. Thus, $\beta(p_1) - p_2 \in \ker \pi_2 = \operatorname{im} i_2$, i.e. $\beta(p_1) - p_2 = i_2(n_2)$ for some $n_2 \in N_2$. Hence, $p_2 = \beta(p_1) - i_2(n_2)$, which implies that g is surjective.

P_2 is projective, so the sequence splits, which is equivalent to $P_1 \oplus N_2 \cong P_2 \oplus N_1$. □

Problem 7. Let $N \in \mathbb{Z}$, $N \geq 2$. Prove that $\mathbb{Z}/N\mathbb{Z}$ is injective as an $\mathbb{Z}/N\mathbb{Z}$ module. (*Warning: the ring under consideration is not a domain, so Corollary 17.2 page 4 does not apply.*)

Proof. We use Baer's criterion to prove that $R = \mathbb{Z}/N\mathbb{Z}$ is injective. The ideals of $\mathbb{Z}/N\mathbb{Z}$ correspond to the ideals in $\mathbb{Z}$ containing N, therefore any ideal $I \subset R$ is of the form (d), $d|N$. Consider the following diagram.

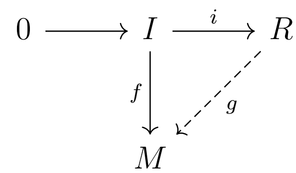
$$\begin{array}{ccccc}
0 & \longrightarrow & I & \xrightarrow{i} & R \\
 & & \Big\downarrow{\scriptstyle f} & \swarrow{\scriptstyle g} & \\
 & & R & &
\end{array}$$

Suppose $f(d) = a$ and $d'd = N$. Then $0 = f(N) = f(d'd) = d'f(d) = d'a$ in $\mathbb{Z}/N\mathbb{Z}$. There exists $x \in \mathbb{Z}$ such that $d'a = Nx = d'dx$. $d' \neq 0$, so $d|a$ as intergers in $\mathbb{Z}$. We can define $g : R \to R$ by $g(1) = a/d = x$. Then $g|_I = f$ and hence R is an injective R-module. □

Problem 8. Give an example of a domain R and an R–module M, such that M is divisible but not injective.

Proof. Let $R = \mathbb{Z}[x]$ and $M = \mathbb{Q}(x)/\mathbb{Z}[x]$. $\mathbb{Q}(x)$ is the fraction field of $\mathbb{Z}[x]$, so it is a divisible $\mathbb{Z}[x]$-module. It is not hard to check that M is also a divisible $\mathbb{Z}[x]$-module. Take an ideal $I = (2, x) \subset R$ and define $f : I \to M$ by $f(2) = [0]$ and $f(x) = \left[\frac{1}{2}\right]$.

Assume M is injective, then we have the following diagram.



Namely, there exists $g : R \to M$ such that $2g(1) = g(2) = [0]$ and $xg(1) = g(x) = \left[\frac{1}{2}\right]$. Suppose $g(1) = a(x) + \mathbb{Z}[x]$, where $a(x) \in \mathbb{Q}(x)$. Then $2a(x) \in \mathbb{Z}[x]$ and $xa(x) - \frac{1}{2} \in \mathbb{Z}[x]$, say $2a(x) = b(x)$ and $xa(x) - \frac{1}{2} = c(x)$. It follows that $2a(0) = b(0) \in \mathbb{Z}$ and therefore $-\frac{1}{2} = 0 \cdot a(0) - \frac{1}{2} = c(0) \in \mathbb{Z}$, a contradiction. Hence, M is not injective. $\square$

Problem 10. For $M, N \in R-\mathbf{mod}$ and $Q \in \mathbf{Ab}$, prove that we have an isomorphism of R–modules:

$$\mathrm{Hom}_R\left(M, \mathrm{Hom}_{\mathbb{Z}}(N, Q)\right) \cong \mathrm{Hom}_{\mathbb{Z}}\left(M \otimes_R N, Q\right)$$

where, recall that, for an R–module X, and an abelian group Y, we defined an R–module structure on $\mathrm{Hom}_{\mathbb{Z}}(X, Y)$ by:

$$(r \cdot \xi)(x) = \xi(rx) \text{ for every } r \in R,\ x \in X,\ \xi \in \mathrm{Hom}_{\mathbb{Z}}(X, Y).$$

Proof. Define φ and ψ as follows.

$$\varphi : \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, Q)) \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, Q), \quad \varphi(f)(m \otimes n) = f(m)(n)$$

$$\psi : \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, Q) \to \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, Q)), \quad \psi(g)(m) = [n \mapsto g(m \otimes n)]$$

We will show that φ and ψ are well-defined and inverses to each other.

$f \in \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, Q))$ and $f(m) \in \mathrm{Hom}_{\mathbb{Z}}(N, Q)$, so we have

$$\begin{aligned} \varphi(f)((m_1 + m_2) \otimes n) =& f(m_1 + m_2)(n) = f(m_1)(n) + f(m_2)(n) \\ =& \varphi(f)(m_1 \otimes n) + \varphi(f)(m_2 \otimes n) \end{aligned}$$

$$\begin{aligned} \varphi(f)(m \otimes (n_1 + n_2)) =& f(m)(n_1 + n_2) = f(m)(n_1) + f(m)(n_2) \\ =& \varphi(f)(m \otimes n_1) + \varphi(f)(m \otimes n_2). \end{aligned}$$

Here we treat M as an R-bimodule, i.e. $rm = mr$. Using the R-module structure on $\mathrm{Hom}_{\mathbb{Z}}(N, Q)$, we get

$$\varphi(f)(mr \otimes n) = f(rm)(n) = (rf(m))(n) = f(m)(rn).$$

Thus, φ is well-defined. Also,

$$\begin{aligned}\psi(g)(rm) =& [n \mapsto g(rm \otimes n)] = [n \mapsto g(m \otimes rn)] \\ =& r[n \mapsto g(m \otimes n)] = r\psi(g)(m),\end{aligned}$$

therefore ψ is well-defined.

Note that

$$\psi \circ \varphi(f) = \psi([m \otimes n \mapsto f(m)(n)]) = [m \mapsto [n \mapsto f(m)(n)]] = f$$

$$\varphi \circ \psi(g) = \varphi([m \mapsto [n \mapsto g(m \otimes n)]]) = [m \otimes n \mapsto g(m \otimes n)] = g.$$

Hence, $\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, Q)) \cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, Q)$. □