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ALGEBRA II

30/30

#8

1.) Using #3 from HW7, we get projective resolution of R/m as

$$0 \longrightarrow R \xrightarrow{\binom{y}{-x}} R \oplus R \xrightarrow{(x \ \ y)} R \longrightarrow 0$$

Apply $- \otimes M$

$$R \otimes M \longrightarrow (R \oplus R) \otimes M \longrightarrow R \otimes M \longrightarrow 0$$

$$\cong$$

$$M \xrightarrow{\varphi} M \oplus M \xrightarrow{\psi} M \longrightarrow 0$$

where for any ~~[illegible]~~ $p, q \in M$

$$\varphi(p) = (0, -tp) \qquad \psi(p, q) = tp$$

$$\ker \psi = \{ (f, g) : tf = 0 \} = \{ (0, g) : g \in M \}$$

$$\operatorname{Im} \varphi = \{ (0, -tf) : f \in M \} = \{ (0, f) : f \in M \}$$

$$\Rightarrow \quad \operatorname{Tor}_1^R (R/m, M) = 0$$

R is an integral domain, so M is flat $\Leftrightarrow$ M is torsion free

but consider $\frac{y}{1} \in R$ and $\frac{t}{1} \in M$ then

$$\frac{y}{1} \cdot \frac{t}{1} = \frac{0 \cdot t}{1} = 0$$

$\Rightarrow$ M has torsion

$\Rightarrow$ M is not flat.

3) Let K be a field and

$R = K[x,y]$ $\qquad$ $M = (x,y)$

then R is an integral domain, M is torsion free

as $\forall f \in R$ $\qquad M \xrightarrow{M_f} M$

$\qquad\qquad\qquad\qquad p \qquad f p$

$f \neq 0$ and $fp = 0$ $\Rightarrow$ $p = 0$ as R is an integral domain.

$m = (x,y)$ is maximal ideal of R

Since R_m is a local Noetherian ring, if M_m is flat R_m-module

$\Rightarrow$ M_m is free using Proposition 23.4 but consider $f: R_m \oplus R_m \xrightarrow{(x\ y)} M_m$

then $-y \oplus x \in \ker f$

$\Rightarrow \ker f \neq 0$

$\Rightarrow M_m$ is not free

Thus, M_m is not flat ✓

Using 2.3.2 $\Rightarrow$ M is not flat.

5) $0 \to R \to K \to K/R$ is a s.e.s.

Apply $- \otimes N$

$$\to \cancel{\mathrm{Tor}_1^R(K,N)}^{\,0} \to \mathrm{Tor}_1^R(K/R, N) \to R \otimes N \to K \otimes N \to K/R \otimes N \to 0$$

K is field of fractions of R $\Rightarrow$ ~~K is flat~~ $\Rightarrow$ $\mathrm{Tor}_1^R(K,N) = 0$

$\Rightarrow \quad \mathrm{Tor}_1^R(K/R, N) = \ker(R \otimes N \longrightarrow K \otimes N)$

we know $\quad R \otimes N \simeq N$

$K \otimes N \simeq S^{-1}N \qquad$ (where $K = S^{-1}R$)

let $n \in \ker(N \longrightarrow S^{-1}N) \iff \frac{m}{1} = 0 \in S^{-1}N$

$\iff \exists\, t \in S$ s.t.

$tn = 0$

$\iff m \in N_{tor}$

$\therefore \quad \mathrm{Tor}_1^R(K/R, N) \simeq N_{tor}$