ALGEBRA 2. PROBLEM SET 1

Unless otherwise stated, the functors below are covariant.

Problem 1. Let $\mathcal{C}$ be a category. Prove that if an initial object exists in $\mathcal{C}$, then it is unique up to unique isomorphism. Prove the analogous statement about final objects.

Problem 2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a category $\mathcal{C}$ to another category $\mathcal{D}$. Assume that F is faithful. For a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, prove that if $F(f)$ is injective (resp. surjective) then so is f.

Problem 3. Give an example of a morphism in a category which admits a section (i.e, right inverse) but is not surjective. Give a similar example for the other case: a morphism which admits a retraction but is not injective.

Problem 4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let f be a morphism in $\mathcal{C}$. Assume that f has a left (resp. right) inverse. Prove that the same is true for $F(f)$. Is the similar assertion true, where instead we have f injective (resp. surjective)?

Problem 5. Let **Gps** be the category of groups. Let $f : G \to H$ be a morphism. Prove that f is injective in **Gps** if, and only if $\mathrm{Ker}(f) = \{g \in G : f(g) = e\} = \{e\}$.

Problem 6. Give an example of a functor which is faithful and dense but not full.

Problem 7. Consider the following category, denoted by $\mathbf{Ab}_f$. Its objects are filetered abelian groups: $G_\bullet = G_0 \supseteq G_1 \supseteq \ldots$ where G_j is an abelian group and G_{j+1} is a subgroup of G_j (for every $j \geq 0$). Morphisms are defined by:

$$\mathrm{Hom}(G_\bullet, H_\bullet) = \{f : G_0 \to H_0 \text{ group homomorphism such that } f(G_j) \subset H_j, \ \forall\, j \geq 0\}$$

Give an example of a morphism in $\mathbf{Ab}_f$ which is a bijection but not an isomorphism.

Problem 8. For a monoid M, let $\underline{M}$ denote the category which has only one object, say p, and $\mathrm{End}_{\underline{M}} = M$. Prove that functors between two such categories $\underline{M}$ and $\underline{N}$ are same as homomorphisms of monoids $F : M \to N$. Verify that natural transformations between two functor F, G correspond to elements $b \in N$ such that $b.F(x) = G(x).b$ for every $x \in M$.

Problem 9. Let G be a group and let $G-$**Sets** be the category whose objects are sets together with a G–action, and morphisms are set maps which commute with the G–action:

Objects. a set X together with a set map $G \times X \to X$ denoted by $(g, x) \mapsto gx$ such that (a) $ex = x$ for every $x \in X$, (b) $g(hx) = (gh)x$ for every $g, h \in G$.

Morphisms. set maps $f : X \to Y$ such that for every $g \in G$ and $x \in X$, we have $gf(x) = f(gx)$.

Prove that natural isomorphisms of the forgetful functor $F : G - \mathbf{Sets} \to \mathbf{Sets}$ are given by elements of the group G. That is, $\mathrm{Aut}(F) = G$.

Problem 10. Recall the definition of the category $\mathbf{Mat}_K$ (Lecture 2, section 2.1) where K is a fixed field. Recall that we have a functor $F : \mathbf{Mat}_K \to \mathbf{Vect}_K^{\mathrm{fd}}$.

(1) Prove that F is an equivalence of categories.

(2) Prove that constructing a functor $G : \mathbf{Vect}_K^{\mathrm{fd}} \to \mathbf{Mat}_K$ together with natural isomorphisms

$$\varphi : \mathrm{Id}_{\mathbf{Mat}_K} \to GF \qquad \text{and} \qquad \psi : \mathrm{Id}_{\mathbf{Vect}_K^{\mathrm{fd}}} \to FG$$

is same as making a choice of a basis $B_G(V)$ for each finite–dimensional vector space V.

(3) Let G_1, G_2 be two such functors, obtained by choosing $B_1(V), B_2(V)$ respectively, two bases of V, for every finite–dimensional vector space V. Prove that the change of basis matrix provides a natural isomorphism between G_1 and G_2.