ALGEBRA 2. PROBLEM SET 3

Problem 1. Let $(I,\leq)$ be a preordered set such that $i \leq j$ for every $i,j \in I$. Let $\mathcal{C}$ be a category and let $\mathfrak{X} = (\{X_i\}_{i\in I}; \{\varphi_{ji}\}_{i\leq j})$ be a direct system on $(I,\leq)$ with values in $\mathcal{C}$. Prove that, for every $i \in I$, $\varinjlim_{(I,\leq)} \mathfrak{X}$ is isomorphic to X_i. Similarly let $\mathfrak{Y} = (\{Y_i\}_{i\in I}; \{\psi_{ij}\}_{i\leq j})$ be an inverse system over $(I,\leq)$ with values in $\mathcal{C}$. Prove that $\varprojlim_{(I,\leq)} \mathfrak{Y}$ is isomorphic to Y_i, for every $i \in I$. (*Hence, for questions regarding direct/inverse limits, we can assume that the preordered set is in fact partially ordered*).

Problem 2. Let $(I,\leq)$ be a partially ordered set with a unique maximal element, say $i_0 \in I$. Again, let $\mathfrak{X}$ be a direct system over $(I,\leq)$ with values in $\mathcal{C}$. Prove that $\varinjlim_{(I,\leq)} \mathfrak{X}$ is isomorphic to X_{i_0}. Similarly, let $\mathfrak{Y}$ be an inverse system. Prove that $\varprojlim_{(I,\leq)} \mathfrak{Y}$ is isomorphic to Y_{i_0}.

Problem 3. Let J be a set and let $\mathcal{C}$ be a category. Assume that for every set of objects $\{X_j\}_{j\in J}$ of $\mathcal{C}$, the direct sum $\bigoplus_{j\in J} X_j$, and the direct product $\prod_{j\in J} X_j$ exist. Let $\mathcal{C}^J$ be the product category and we view $\bigoplus$ and $\prod$ as functors:

$$\mathcal{C}^J \underset{\prod}{\overset{\bigoplus}{\rightrightarrows}} \mathcal{C}$$

Let $F : \mathcal{C} \to \mathcal{C}^J$ be the canonical inclusion functor. That is, $F(X)_j = X$ for every $j \in J$ and $X \in \mathcal{C}$; and for every morphism $f : X \to Y$ in $\mathcal{C}$, $F(f)_j = f$ for every $j \in J$. Prove that the following two pairs are adjoint functors:

$$\left(\bigoplus\ ,\ F\right) \qquad \text{and} \qquad \left(F\ ,\ \prod\right)$$

Problem 4. Let $I = \{1,2,3,\ldots\}$ together with the usual order. Consider the following inverse system on I valued in **Ab**, denoted by $\mathfrak{Z} = (\{Z_n\}_{n\in I}; \{\varphi_{nm}\}_{n\leq m})$:

- $Z_n = \mathbb{Z}$ for every $n \in I$.
- For every $n \leq m$, the group homomorphism $\varphi_{nm} : Z_m \to Z_n$ is given by $\varphi_{nm}(x) = 3^{m-n}x$.

Prove that $\varprojlim_{(I,\leq)} \mathfrak{Z} = (0)$.

Problem 5. Given a preordered set $(I,\leq)$, consider the category, $\mathrm{Cat}(I,\leq)$, whose objects are elements of I and for every $i,j \in I$, $\mathrm{Hom}_{\mathrm{Cat}(I,\leq)}(i,j)$ is a singleton, say $\{a(i,j)\}$, if $i \leq j$; and is empty if $i \not\leq j$. Verify that a direct (resp. inverse) system on $(I,\leq)$, valued in a category $\mathcal{C}$, is an object of the category of covariant (resp. contravariant) functors from $\mathrm{Cat}(I,\leq)$ to $\mathcal{C}$. *Hence, we have a category of direct (resp. inverse) systems on $(I,\leq)$ valued in $\mathcal{C}$, denoted as follows:*

$$\mathcal{C}^{(I,\leq)} := \mathbf{Func}\,(\mathrm{Cat}(I,\leq),\mathcal{C}) \ \text{ category of direct systems}$$
$$\mathcal{C}_{(I,\leq)} := \mathbf{Func}\,(\mathrm{Cat}(I,\leq)^{\mathrm{op}},\mathcal{C}) \ \text{ category of inverse systems}$$

Problem 6. Let $\mathcal{C}$ be a category and $(I,\leq)$ be a preordered set. For each $X \in \mathcal{C}$, we can define a direct system $\mathfrak{X}$ where each $X_i = X$ and each $\psi_{ji} = \mathrm{Id}_X$. Prove that the direct limit (if exists) can be interpreted as a functor $\mathcal{C}^{(I,\leq)} \to \mathcal{C}$, and it is adjoint (left or right?) to the functor thus obtained $F : \mathcal{C} \to \mathcal{C}^{(I,\leq)}$. Obtain a similar assertion for the inverse limit.

Problem 7. Consider the inverse system of Problem 4 above, denoted there by $\mathfrak{Z}$. Let $\mathfrak{Z}'$ be another inverse system over the same ordered set $I = \{1, 2, 3, \ldots\}$ valued in **Ab**, where each $Z'_n = \mathbb{Z}/2\mathbb{Z}$ and all the morphisms are identities. Prove that the natural surjection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is a morphism in the category of inverse systems $\mathbf{Ab}_{(I,\leq)}$. (*This is one of the standard examples of the fact that inverse limit of surjections need not be a surjection.*).

Problem 8. Prove that the inverse limit of injective morphisms is injective. Prove that the direct limit of surjective morphisms is surjective.

Problem 9. Let $\mathfrak{X}$ be a direct system over a preordered set $(I, \leq)$ valued in **Sets**. Let $X := \sqcup_{i \in I} X_i / \sim$ where the equivalence relation is:

$$x \in X_i \sim \psi_{ji}(x) \in X_j \text{ for every } i \leq j$$

Prove that X is isomorphic to $\varinjlim_{(I,\leq)} \mathfrak{X}$.

Problem 10. With the set up of Problem 9 above, assume that I is *right directed* and each X_i has a structure of a group and each ψ_{ji} is a group homomorphism. Prove that X has a natural structure of a group which makes it isomorphic to the direct limit of $\mathfrak{X}$ in the category of groups.