ALGEBRA 2. PROBLEM SET 4

Problem 1. Let $\mathcal{C}$ be an arbitrary category and let $\mathcal{A}$ be an abelian category. Prove that $\mathbf{Func}\,(\mathcal{C},\mathcal{A})$ (and similarly $\mathbf{Func}\,(\mathcal{C}^{\mathrm{op}},\mathcal{A})$) is again an abelian category.

Remark.– *Many examples of abelian categories arise in this way. For example, categories of direct/inverse systems valued in $\mathcal{A}$ (see Problem 5 of Set 3); presheaves of abelian groups over a topological space; category of complexes over $\mathcal{A}$. This one problem shows us how they are all abelian.*

Problem 2. Let $\mathcal{C}$ be an additive category and let $\{X_i\}_{i\in I}$ be a set of objects of $\mathcal{C}$ such that their direct product exists in $\mathcal{C}$. Prove that, for every $Y \in \mathcal{C}$ we have isomorphisms (of abelian groups)

$$\mathrm{Hom}_{\mathcal{C}}\left(\prod_{i\in I} X_i, Y\right) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$$

where on the right–hand side, the direct sum is that of abelian groups.

Problem 3. Let $\mathcal{C}$ be an additive category such that for any set I and a set of objects $\{X_i\}_{i\in I}$, both $\oplus_{i\in I} X_i$ and $\prod_{i\in I} X_i$ exist in $\mathcal{C}$. Prove that there is a natural transformation α of functors:

$$\begin{tikzcd} \mathcal{C}^I \arrow[r, bend left, "\bigoplus"] \arrow[r, bend right, "\prod"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \end{tikzcd}$$

Problem 4. Assume $\mathcal{C}$ is an additive category where arbitrary direct sums and products exist. Further assume that direct sums and products are always isomorphic. Prove that every object in $\mathcal{C}$ is a zero object.

Problem 5. Let $\mathcal{C}$ be an additive category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Write down a functor $\mathcal{C} \to \mathbf{Ab}$ whose representability is equivalent to the existence of a kernel of f in $\mathcal{C}$ (the functor, if representable, will be represented by the kernel). Do the same exercise for the cokernel.

Problem 6. Let I be a set and let $\mathbb{Z}^{(I)}$ be the direct sum of abelian groups $\{X_i = \mathbb{Z}\}_{i\in I}$. Prove that $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}^{(I)}, -)$ is an exact functor $\mathbf{Ab} \to \mathbf{Ab}$.

In the problems below R is a non–zero ring with 1.
And $R-\mathbf{mod}$ is the category of left R–modules.

Problem 7. Let J be a set. Prove that $\bigoplus_J$ and $\prod_J$ are exact functors $R\text{–}\mathbf{mod}^J \to R\text{–}\mathbf{mod}$. Here $R\text{–}\mathbf{mod}^J$ is the product category (it is abelian by Problem 1 above).

Problem 8. Let $(I,\leq)$ be a right directed preordered set. Prove that $\varinjlim_{(I,\leq)} : R-\mathbf{mod}^{(I,\leq)} \to R-\mathbf{mod}$ is an exact functor. Here, $R-\mathbf{mod}^{(I,\leq)}$ is the category of directed systems over $(I,\leq)$ with values in the category $R-\mathbf{mod}$ (defined in Problem 5 of Set 3, and is abelian by Problem 1 above).

Problem 9. Let $(I,\leq)$ be a preordered set. Prove that $\varprojlim_{(I,\leq)} : R-\mathbf{mod}_{(I,\leq)} \to R-\mathbf{mod}$ is left exact, but not right exact.