ALGEBRA 2. PROBLEM SET 5

In all problems below, $\mathcal{A} = R-\mathbf{mod}$ is the category of modules over a commutative ring R (with $1 \neq 0$). We denote by $\mathbb{K}^\bullet(\mathcal{A})$ the category of cochain complexes of $\mathcal{A}$.

Problem 1. Consider the following diagram in $\mathcal{A}$ with exact rows and commuting squares.

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "f"'] & N \arrow[r, "v"] \arrow[d, "g"'] & P \arrow[d, "h"'] \\ M' \arrow[r, "u'"'] & N' \arrow[r, "v'"'] & P' \end{tikzcd}$$

(1) Prove that, if u', f, h are injective, then g is injective.

(2) Prove that, if v, f, h are surjective, then g is surjective.

(3) Prove that, if g is injective and f, v are surjective, then h is injective.

(4) Prove that, if g is surjective and h, u' are injective, then f is surjective.

Problem 2. Prove the analogue of Theorem 16.3, page 3-4, for projective resolutions.

Problem 3. Consider the subcategory of $\mathbb{K}^\bullet(\mathcal{A})$ consisting of exact complexes (i.e, $C^\bullet$ such that $H^n(C^\bullet) = 0$ for every $n \in \mathbb{Z}$). Give an example to illustrate that this subcategory is not closed under taking kernels.

Problem 4. *Five lemma.* Consider the following commutative diagram with exact rows.

$$\begin{tikzcd} M_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & M_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & M_3 \arrow[r, "a_3"] \arrow[d, "f_3"'] & M_4 \arrow[r, "a_4"] \arrow[d, "f_4"'] & M_5 \arrow[d, "f_5"'] \\ N_1 \arrow[r, "b_1"'] & N_2 \arrow[r, "b_2"'] & N_3 \arrow[r, "b_3"'] & N_4 \arrow[r, "b_4"'] & N_5 \end{tikzcd}$$

(1) Prove that if f_2, f_4 are injective and f_1 is surjective, then f_3 is injective.

(2) Prove that if f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.

Hence, if f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Problem 5. Let $C^\bullet$ and $D^\bullet$ be two cochain complexes of R–modules. Let $\alpha^\bullet : C^\bullet \to D^\bullet$ be a morphism. Define:

- $C[1]^\bullet \in \mathbb{K}^\bullet(\mathcal{A})$ is a complex with $C[1]^n = C^{n+1}$ and $d^n_{C[1]} = -d^{n+1}_C$, for every $n \in \mathbb{Z}$.
- $\mathrm{Cone}(\alpha)$ is the following complex:
  - For every $n \in \mathbb{Z}$, $\mathrm{Cone}(\alpha)^n = C^{n+1} \oplus D^n$.
  - For every $n \in \mathbb{Z}$, the differential, denoted by D^n below, is given by the following expression, for every $x \in C^{n+1}$ and $y \in D^n$,
  $$D^n(x, y) = (-d^{n+1}_C(x), d^n_D(y) - \alpha^{n+1}(x))$$

(1) Verify that $\mathrm{Cone}(\alpha)$ is a complex (i.e, $D \circ D = 0$).

(2) Prove that we have the following short exact sequence of cochain complexes.
$$0 \to D^\bullet \to \mathrm{Cone}(\alpha) \to C^\bullet[1] \to 0$$

(3) Prove that $\alpha^\bullet$ is null homotopic if, and only if, the sequence above is split.

(4) Prove that $H^n(\alpha)$ is an isomorphism, for every $n \in \mathbb{Z}$ if, and only if $H^n(\mathrm{Cone}(\alpha)) = 0$ for every $n \in \mathbb{Z}$.