ALGEBRA 2. PROBLEM SET 6

In all problems below, R is a commutative ring (with $1 \neq 0$).
*$R-$**mod** is the category of R–modules.*

Problem 1. Let $P \in R-\mathbf{mod}$. Prove that P is projective if, and only if, there exists $P' \in R-\mathbf{mod}$ such that $P \oplus P'$ is a free R–module.

Problem 2. Let $p \in R$ be such that $p^2 = p$. Prove that the ideal Rp generated by p is a projective R–module.

Problem 3. For a $P \in R-\mathbf{mod}$, prove that the following conditions are equivalent.

(1) P is projective.
(2) For every $M \in R-\mathbf{mod}$ and $k \geq 1$, $\mathrm{Ext}_R^k(P, M) = 0$.
(3) For every $M \in R-\mathbf{mod}$, $\mathrm{Ext}^1(P, M) = 0$.

Problem 4. Consider the following two short exact sequences of R–modules, where P_1 and P_2 are projective.

$$0 \longrightarrow N_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow N_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0$$

Prove that $P_1 \oplus N_2$ is isomorphic to $P_2 \oplus N_1$.

Problem 5. Let $R = K[x, y, z]$ be the polynomial ring in three variables over a field K, and let $M = K$ with the natural R–action (see Lecture 18, pages 5–6). Prove that the sequence of morphisms written at the end of Lecture 18 page 6, is a free resolution of the R–module M.

Problem 6. Prove that an arbitrary direct sum of projective modules is projective. Prove that an arbitrary direct product of injective modules is injective.

Problem 7. Let $N \in \mathbb{Z}$, $N \geq 2$. Prove that $\mathbb{Z}/N\mathbb{Z}$ is injective as an $\mathbb{Z}/N\mathbb{Z}$ module. (*Warning: the ring under consideration is not a domain, so Corollary 17.2 page 4 does not apply.*)

Problem 8. Give an example of a domain R and an R–module M, such that M is divisible but not injective.

Problem 9. Assume that R is injective as an R–module, and a domain. Prove that R is a field.

Problem 10. For $M, N \in R-\mathbf{mod}$ and $Q \in \mathbf{Ab}$, prove that we have an isomorphism of R–modules:

$$\mathrm{Hom}_R\left(M, \mathrm{Hom}_{\mathbb{Z}}(N, Q)\right) \cong \mathrm{Hom}_{\mathbb{Z}}\left(M \otimes_R N, Q\right)$$

where, recall that, for an R–module X, and an abelian group Y, we defined an R–module structure on $\mathrm{Hom}_{\mathbb{Z}}(X, Y)$ by:

$$(r \cdot \xi)(x) = \xi(rx) \text{ for every } r \in R,\ x \in X,\ \xi \in \mathrm{Hom}_{\mathbb{Z}}(X, Y).$$