ALGEBRA 2. PROBLEM SET 7

In all problems below, R is a commutative ring (with $1 \neq 0$).
*$R-$**mod** is the category of R–modules.*

The problem below is given to review some of the results from Algbera I. You can consult, for example, Lectures 30, 31 of https://people.math.osu.edu/gautam.42/F17/algebra.html

Problem R. Let $S \subset R$, $1 \in S$, $0 \notin S$ be a multiplicatively closed set. Let M be an R–module.

(1) For every ideal $\mathfrak{a} \subset R$, prove that we have an isomorphism: $(R/\mathfrak{a}) \otimes_R M \cong M/\mathfrak{a}M$.

(2) Prove that $S^{-1}R \otimes_R M \cong S^{-1}M$.

(3) Prove that for every short exact sequence of R–modules: $0 \to M_1 \to M_2 \to M_3 \to 0$, the following sequence is again exact.

$$0 \to S^{-1}M_1 \to S^{-1}M_2 \to S^{-1}M_3 \to 0$$

Problem 1. Let R be a principal ideal domain and let $a \in R$, $a \neq 0$. Prove that, for every R–module N, we have an isomorphism $\mathrm{Ext}^1_R(R/(a), N) \cong N/aN$.

Problem 2. Let R be an integral domain. Let K be its field of fractions. In this problem, we view a K–vector space V as an R–module, via restricting the scalars $R \subset K$.

(1) For an R–module M, let M_{tor} be defined as the set of torsion elements of M:

$$M_{\mathrm{tor}} = \{x \in M : a.x = 0 \text{ for some non–zero } a \in R\}$$

Prove that $M_{\mathrm{tor}} \otimes_R K = \{0\}$, and $M \otimes_R K \cong (M/M_{\mathrm{tor}}) \otimes_R K$.

(2) Let M be a torsion–free R–module. Prove that the natural map $M = M \otimes_R R \to M \otimes_R K$ is injective. (*Hence, M is an R–submodule of a K vector space.*). If M is both torsion–free and divisible, prove that this map is an isomorphism.

(3) Prove that every K–vector space V is a flat R–module.

Problem 3. Let R be an arbitrary commutative ring. Assume that there is a non–zero divisor $a \in R$; and $b \in R$ which is a non–zero divisor in $R/(a)$. Prove that the following is a projective resolution of $M = R/(a, b)$.

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} b \\ -a \end{pmatrix}} R \oplus R \xrightarrow{(a \;\; b)} R \longrightarrow 0$$

Problem 4. Use Problem 3, to compute $\mathrm{Ext}^\bullet_R(M, N)$ for the following cases.

(1) $R = \mathbb{Z}[x]$, $M = N = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}[x]/(2, x)$.

Bonus. For each element of $\operatorname{Ext}^1_R(M, N)$, with R, M, N as in the line above, write the corresponding short exact sequence of $\mathbb{Z}[x]$–modules: $0 \to N \to X \to M \to 0$.

(2) $R = K[x, y]$ where K is any field, and $M = N = K$.

Problem 5. Let $R = \mathbb{Z}/4\mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$ with the natural R–action. Compute $\operatorname{Ext}^\bullet_R(M, M)$.

Problem 6. Assume that we have a short exact sequence of R–modules, where F is a flat R–module: $0 \to N \to M \to F \to 0$. Prove that, for every R–module E, the following sequence is exact: $0 \to N \otimes E \to M \otimes E \to F \otimes E \to 0$.

Hint for problem 6: take a projective module P, with a surjective map $P \to E$. Let K be the kernel of this surjection. Draw a snake lemma–style cartoon where you tensor the given short exact sequence with K for the top row and P for the bottom row.

Problem 7. Again, assume that there is a short exact sequence of R–modules, where F is a flat R–module: $0 \to F_1 \to F_2 \to F \to 0$. Prove that F_1 is flat if, and only if F_2 is flat.

Hint for problem 7: any given injective morphism $A \to B$ will give rise to a snake–lemma–type picture, when tensored with the given short exact sequence, just like in problem 6 above.