ALGEBRA 2. PROBLEM SET 8

In all problems below, R is a commutative ring (with $1 \neq 0$).

Problem R. Let $S \subset R$, $1 \in S$, $0 \notin S$ be a multiplicatively closed set. Let M, N be two R–modules. Prove that we have the following isomorphisms of $S^{-1}R$–modules.

$$S^{-1}M \otimes_{S^{-1}R} S^{-1}N \cong S^{-1}M \otimes_R N \cong S^{-1}(M \otimes_R N)$$

Formulate and prove the similar assertion for Hom.

Problem 1. Let K be a field and let R be the localization of the polynomial ring $K[x, y]$ at the ideal $(x, y) \subset K[x, y]$. Let $\mathfrak{m}$ denote the unique maximal ideal of R. Take $M = K(t)$ (rational functions in one variable t) with R–action:

$$\left(\frac{f(x,y)}{g(x,y)}\right) \cdot \frac{p(t)}{q(t)} = \frac{f(t,0)}{g(t,0)} \frac{p(t)}{q(t)}$$

where $f(x, y), g(x, y) \in K[x, y]$ are such that $g(0, 0) \neq 0$, and $p(t), q(t) \in K[t]$ are such that $q(t) \neq 0$.

Prove that $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$ and that M is not flat.

Problem 2. Let R be a local principal ideal domain. Let M be a finitely–generated R–module. Prove that $M \cong M_{\mathrm{tor}} \oplus (M/M_{\mathrm{tor}})$.

Problem 3. Prove or disprove: every torsion–free module over an integral domain is flat.

Problem 4. Let M, N be two flat R–modules. Prove that so is $M \otimes_R N$.

Problem 5. Let R be an integral domain and K be its field of fractions. For any R–module, N, prove that $\mathrm{Tor}_1^R(K/R, N) = N_{\mathrm{tor}}$.

Problem 6. Let k be a field and consider $R = k[x, y]/(y^2 - x^3)$. Let $R' = k[t]$ with a structure of an R–module via the ring homomorphism $x \mapsto t^2$ and $y \mapsto t^3$. Prove or disprove: R' is a flat R–module.