ALGEBRA 2. PROBLEM SET 10

Problem 1. Let E/F be an algebraic extension. Let $\sigma : E \to E$ be a morphism of fields such that $\sigma|_F = \mathrm{Id}_F$. Prove that σ is then an isomorphism.

Problem 2. Give an example to prove that the assertion of Problem 1 is false, if E/F is not assumed to be algebraic.

Problem 3. Let E/F be a normal extension. Let K be an intermediate field: $F \subset K \subset E$ and let $\sigma : K \to E$ be an embedding which is the identity on F. Prove that σ extends to an automorphism of E.

Problem 4. Let $F = \mathbb{Q}$, E be the splitting field of $X^2 - 2 \in \mathbb{Q}[X]$ over $\mathbb{Q}$. Thus $E = \mathbb{Q}(\sqrt{2})$. Let K be the splitting field of $X^2 - \sqrt{2} \in E[X]$ over E. Describe the group $\mathcal{G}al(K/F)$. Prove that K/F is not a Galois extension.

Problem 5. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $F = \mathbb{Q}$. Compute $\mathcal{G}al(E/F)$. Is E/F a Galois extension?

Problem 6. Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{R}$, and let $E = \mathbb{Q}(\alpha)$ (with $F = \mathbb{Q}$). Compute $\mathcal{G}al(E/F)$.

Problem 7. For a finite extension E/F, prove that $\mathcal{G}al(E/F)$ is finite.

Problem 8. Let $\zeta = \exp\left(\frac{2\pi\iota}{7}\right) \in \mathbb{C}$. Prove that $\mathbb{Q}(\zeta)$ is a Galois extension of $\mathbb{Q}$.

Problem 9. Let $k \subset K$ be two fields, and assume that K is algebraically closed. Let K' consist of all elements of K which are algebraic over k. Prove that K' is isomorphic to the algebraic closure of k.

Problem 10. Consider the field $E = \mathbb{Q}(T_1, \ldots, T_n)$. For each $k \in \{1, \ldots, n\}$ define:

$$\varepsilon_k := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} T_{i_1} T_{i_2} \cdots T_{i_k}$$

and let $F = \mathbb{Q}(\varepsilon_1, \ldots, \varepsilon_n) \subset E$. Prove that $(E : F) = n!$ in the following steps.

(1) Prove that E/F is the splitting extension of a degree n polynomial. (*Hence by Problem 3 of Problem Set 9, $(E : F)$ divides $n!$.*)

(2) Let S_n be the permutation group on n letters, acting on E in the obvious way. Prove that $F \subset E^{S_n}$. (*Hence, $(E : F) \ge (E : E^{S_n}) = n!$ by Artin's theorem.*)