ALGEBRA 2. PROBLEM SET 11

Problem 1. Let K/F be a finite Galois extension. Let E_ℓ be sub-F-extensions of K ($\ell = 1, 2$) and let $H_\ell = \mathcal{G}al\,(K/E_\ell)$ be the corresponding subgroups of $G = \mathcal{G}al\,(K/F)$. Let E be the smallest sub-F-extension of K containing both E_1 and E_2.

(1) Prove that $\mathcal{G}al\,(K/E) = H_1 \cap H_2$.

(2) Prove that $\mathcal{G}al\,(K/E_1 \cap E_2) = H$ is the subgroup generated by H_1 and H_2.

(3) Prove that $E_1 \subset E_2$ if, and only if $H_1 \supset H_2$.

Problem 2. Retain the notations of Problem 1, and further assume that $K = E$ and $E_1 \cap E_2 = F$.

(1) Assuming E_1/F is normal, prove that G is the semi–direct product $H_1 \rtimes H_2$.

(2) Assuming both E_ℓ/F are normal, prove that $G = H_1 \times H_2$.

Problem 3. Let $p \in \mathbb{Z}_{\geq 2}$ be a prime and $q = p^r$. Set $F = \mathbb{F}_q$ the finite field with exactly q elements. Let E/F be a finite extension of degree m.

(1) Prove that $\sigma_q : E \to E$ $(a \mapsto a^q)$ is an element of $\mathcal{G}al\,(E/F)$.

(2) Prove that σ_q has order exactly m.

(3) Prove that $\mathcal{G}al\,(E/F)$ is generated by σ_q, and that E/F is a Galois extension.

Problem 4. Let E/F be an algebraic extension and assume that F is perfect. Prove that E is perfect.

Problem 5. Again let $p \in \mathbb{Z}_{\geq 2}$ be a prime. Consider the imperfect field $F = \mathbb{F}_p(\lambda)$. Inductively, define fields $F_j = \mathbb{F}_p(\lambda_j)$ where $F_0 = F$, $\lambda_0 = \lambda$, and F_{j+1} is the splitting extension of $X^p - \lambda_j \in F_j[X]$. (*Thus it is generated by one element λ_{j+1} such that $\lambda_{j+1}^p = \lambda_j$.*) Set $E = \bigcup_{j \geq 0} F_j$. Prove or disprove that E is perfect. Also, prove that $\mathcal{G}al\,(E/F)$ is trivial.

Problem 6. Let E/F be a finite Galois extension. Let $\alpha \in E$. Prove that $E \cong F(\alpha)$ if, and only if $\mathrm{Stab}_G(\alpha) = \{e\}$ where $G = \mathcal{G}al\,(E/F)$.

Problem 7. Let $f(X) = X^3 - aX + b \in \mathbb{Q}[X]$. Let $E/\mathbb{Q}$ be the splitting extension of $f(X)$. (*Recall that $(E : \mathbb{Q})$ divides* 6.) Let $\alpha, \beta, \gamma \in E$ be three roots of $f(X)$ and set

$$\delta := (\alpha - \beta)(\beta - \gamma)(\alpha - \gamma) \qquad \Delta := \delta^2$$

(1) Prove that $\Delta \in \mathbb{Q}$. *The explicit formula is $\Delta = 4a^3 - 27b^2$. But you don't have to prove this, even though it is hours of fun. Δ, as a polynomial in the coefficients of f, is called the discriminant of f.*

(2) Prove that $\mathcal{G}al\,(E/\mathbb{Q})$ is S_3 if, and only if $\delta \notin \mathbb{Q}$. That is, Δ is not a complete square in $\mathbb{Q}$.