Lecture 0.

(0.0) Three main topics to be covered in this course

Category Theory	Homological Algebra	Galois Theory
Mainly as a convenient language for second part	Category R-mod Functors - Ext, Tor	Fields and extensions (little overlap with the rest)

(i) Categories – introduced by Eilenberg and MacLane (around 1945) as a "convenient language".

(ii) Ext (short for extensions) was the first "functor" studied by Eilenberg and MacLane. Today most of cohomology theories are Ext's in an appropriate category.

(iii) Tor functors are heavily used in intersection theory (e.g. Serre's famous intersection mult. formula).

(0.1) Category. A category $\mathcal{C}$ consists of

- a <u>class</u> $Ob(\mathcal{C})$ (called objects of $\mathcal{C}$)
- for any two $X, Y \in Ob(\mathcal{C})$, a <u>set</u> $Hom_{\mathcal{C}}(X,Y)$ (called morphisms of $\mathcal{C}$)

- for any $X, Y, Z \in Ob(\mathcal{C})$, a set map (called composition)

$$Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \longrightarrow Hom_{\mathcal{C}}(X,Z)$$
$$(f,g) \longmapsto g \circ f$$

- for every $X \in Ob(\mathcal{C})$, an element $Id_X \in Hom_{\mathcal{C}}(X,X)$ (called identity morphism).

This data is subject to following axioms.

- Composition is associative : $h \circ (g \circ f) = (h \circ g) \circ f$
  (for any $f \in Hom_{\mathcal{C}}(X,Y)$ $g \in Hom_{\mathcal{C}}(Y,Z)$ $h \in Hom_{\mathcal{C}}(Z,W)$)
- Identity. $f \circ Id_X = f = Id_Y \circ f$ $\forall f \in Hom_{\mathcal{C}}(X,Y)$

(0.2) Abuse of notation.

(i) Sometimes we just write $X \in \mathcal{C}$ instead of $X \in Ob(\mathcal{C})$.

(ii) $f \in Hom_{\mathcal{C}}(X,Y)$ is usually written as
$$f : X \longrightarrow Y \quad \text{or} \quad X \xrightarrow{f} Y$$

(0.3) Some examples.

(i) <u>Sets</u>. Objects are sets. Morphisms are functions between sets.

(ii) <u>Groups</u> ; <u>Rings</u> ; Commutative Rings etc.
are categories defined analogously.

(iii) Let M be a monoid (i.e. a set together with an associative law of composition and a unit element)
Define $\underline{M}$ to be a category with only one object, say $*$,
and $\mathrm{Hom}_{\underline{M}}(*, *) = M$ ($\mathrm{Id}_* =$ unit of M).

(0.4) Some adjectives associated to morphisms in a category $\mathcal{C}$. -

Let $f : X \to Y$ be a morphism in $\mathcal{C}$.

f is said to be <u>injective</u> (or <u>monomorphism</u>) if $\forall Z \in \mathcal{C}$

$$\begin{aligned} \mathrm{Hom}(Z, X) &\longrightarrow \mathrm{Hom}(Z, Y) \\ g &\longmapsto f \circ g \end{aligned}$$ is one to one.

[That is, $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$]

f is said to be <u>surjective</u> (or <u>epimorphism</u>) if $\forall Z \in \mathcal{C}$

$$\begin{aligned} \mathrm{Hom}(Y, Z) &\longrightarrow \mathrm{Hom}(X, Z) \\ h &\longmapsto h \circ f \end{aligned}$$ is one to one

[That is, $h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$]

f admits a <u>left inverse</u> (also called <u>retraction</u>) if $\exists\, r \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$

s.t. $\quad r \circ f = \mathrm{Id}_X$

f admits a <u>right inverse</u> (also called <u>section</u>) if $\exists\, s \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$

s.t. $\quad f \circ s = \mathrm{Id}_Y$

f is said to be <u>bijective</u> if it is both injective & surjective.

f is said to be an <u>isomorphism</u> if it has both left & right inverses (which are then equal to each other) (& unique).

(0.5) Some easy implications of the definitions.

(i) $g \circ f$ injective $\Rightarrow$ f is injective. As Id_X is injective we obtain: f admits a left inverse $\Rightarrow$ f is injective.

[Proof. $X \xrightarrow{f} Y$. Let $g_1, g_2 : Z \to X$ be such that $f \circ g_1 = f \circ g_2$. Then $g \circ f \circ g_1 = g \circ f \circ g_2$ and by injectivity of $g \circ f$, we get $g_1 = g_2$. Hence, f is injective. $\square$]

(ii) $f \circ h$ surjective $\Rightarrow$ f is surjective. [Similar proof].

As before, we get: f admits a right inverse $\Rightarrow$ f is surjective.

(iii) Isomorphism $\Rightarrow$ Bijective (combining (i) & (ii) above).

(0.6) Counterexamples.

(i) Left and right inverses are not unique. However if f admits both then they are, and equal to each other, usually denoted by f^{-1}.

e.g. Let Vect_K be the category of K vector spaces, where K is a field.

$$f : K^2 \longrightarrow K \qquad (x, y) \longmapsto x+y$$

admits sections (right inverses).

$$S_1 : K \longrightarrow K^2 \qquad x \longmapsto (x, 0)$$

$$S_2 : K \longrightarrow K^2 \qquad x \longmapsto (0, x)$$

$$f \circ S_1 = f \circ S_2 = \mathrm{Id}_K.$$

(ii) In $\mathcal{C}$ = <u>Sets</u>, injectivity = usual one-one. surjectivity = usual onto [Prove This !!]

More generally in a category where objects are <u>sets</u> together with some additional structure (eg. <u>Groups</u>, <u>Rings</u>, <u>Top Spaces</u> ..)

usual one-one $\Rightarrow$ injectivity

usual onto $\Rightarrow$ surjectivity

[argue by elements of the sets $X, Y, Z \in Ob(\mathcal{C})$]

The converse <u>may not hold</u>. Consider $\mathcal{C}$ = <u>Rings</u>.

Let $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion. We claim that f is <u>surjective</u> (but not <u>usual onto</u>). So, let R be a ring and $\mathbb{Q} \underset{h_2}{\overset{h_1}{\rightrightarrows}} R$ two ring homomorphisms, such that $h_1\big|_{\mathbb{Z}} = h_2\big|_{\mathbb{Z}}$ (i.e. $h_1 \circ f = h_2 \circ f$).

As $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$, a ring hom $\mathbb{Q} \to R$ is completely determined by its restriction to $\mathbb{Z}$. Hence, $h_1 = h_2$.

(iii) A bijection that is not an isomorphism

[$\mathbb{Z} \hookrightarrow \mathbb{Q}$ in <u>Rings</u> is a bijection but not an iso.]

Another example Filtered Abelian Groups $=: \mathcal{C}$

Objects : $F_0 \supset F_1 \supset \ldots$ where each F_j is an abelian group.

Morphisms : $\mathrm{Hom}_{\mathcal{C}}(F_\cdot, G_\cdot) = \{$ gp. hom. $\varphi : F_0 \to G_0$ s.t. $\varphi\big|_{F_j} : F_j \to G_j \ \forall j \geq 0 \}$

Take
$$\begin{array}{ccccccc} F_0 & = & \mathbb{Z} & \xrightarrow{\text{Id}} & \mathbb{Z} & = & G_0 \\ & & \cup & & \cup & & \\ F_1 & = & (0) & & 2\mathbb{Z} & = & G_1 \\ & & \cup & & \cup & & \\ & & \vdots & & (0) & & \\ & & & & \cup & & \\ & & & & \vdots & & \end{array}$$
is a bijection but has no inverse.

(0.7) Some adjectives associated with objects of $\mathcal{C}$. – Let $X \in \mathcal{C}$.

A <u>subobject</u> of X is (X', i) where $i : X' \to X$ is an injective morphism in $\mathcal{C}$. There is a partial order: if (X_1, i_1) and (X_2, i_2) are two subobjects of X, we say $(X_1, i_1) \leq (X_2, i_2)$ if $\exists\, j : X_1 \to X_2$ s.t.

$$\begin{tikzcd} X_1 \arrow[d, "j"'] \arrow[dr, "i_1"] \arrow[dr, phantom, "\circlearrowleft" near start] & \\ X_2 \arrow[r, "i_2"'] & X \end{tikzcd}$$

$(i_1 = i_2 \circ j)$. As i_1 is injective, so is j.

$(X_1, i_1) \sim (X_2, i_2)$ if $(X_1, i_1) \leq (X_2, i_2)$ and $(X_2, i_2) \leq (X_1, i_1)$.

[That is,

$$\begin{tikzcd} X_1 \arrow[d, shift right, "\exists j"'] \arrow[dr, "i_1"] & \\ X_2 \arrow[u, shift right, "\exists k"'] \arrow[r, "i_2"'] & X \end{tikzcd}$$

; $i_1 = i_2 \circ j$ and $i_2 = i_1 \circ k$.

In this case, j and k are inverses of each other:

$$i_1 \circ \mathrm{Id}_{X_1} = i_1 = i_2 \circ j = i_1 \circ (k \circ j) \Rightarrow \mathrm{Id}_{X_1} = k \circ j$$

(as i_1 is injective)

Similarly $j \circ k = \mathrm{Id}_{X_2}$.]

It is often convenient to consider subobjects up to ~~isomorp~~ equivalence.

Ex. Formulate the analogous notion of <u>quotient objects</u>.

<u>Initial object</u>. $X \in \mathcal{C}$ is called an initial object if $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is singleton $\forall\, Y \in \mathcal{C}$.

<u>Terminal</u> (or <u>Final</u>) <u>object</u>. $X \in \mathcal{C}$ is called a terminal object if $\operatorname{Hom}_{\mathcal{C}}(Z, X)$ is singleton $\forall\, Z \in \mathcal{C}$.

Easy exercise. Initial object (if exists) is unique up to unique isomorphism. Same for terminal objects.

Null (or Zero) object is an object which is both initial and terminal, usually just denoted by 0.

Examples. (i) $\mathcal{C} =$ <u>Sets</u> has
- initial object (= empty set)
- terminal object (= one element set)
- no null objects

(ii) $\mathcal{C} =$ <u>Groups</u> $\ni$ Trivial group $\{e\}$ is the null object.

(iii) Let K be a field and let $\mathcal{C} =$ category of (f.g.) K-algebras

(i.e. Objects are quotients of polynomial rings $K[x_1, \dots, x_n]$
Morphisms are ring homomorphisms which are identity on K)

$\mathcal{C}$ has an initial object, namely K itself, but no terminal object.