Lecture 1

(1.0) Recall: a category $\mathcal{C}$ consists of (i) a class of objects

(ii) $\forall\ X, Y \in \mathcal{C}$, a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (iii) composition maps

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z) \qquad \text{(iv) } \mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$$

$$(f,g) \longmapsto g\circ f$$

such that
- composition is associative
- Id_X is a unit (neutral element) for composition.

Remark.— In a category $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,X)$ $(=: \mathrm{End}_{\mathcal{C}}(X))$ is always a monoid with respect to composition.

(1.1) Functors.— Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A (<u>covariant</u>) <u>functor</u> F from $\mathcal{C}$ to $\mathcal{D}$ is a rule that assigns

(i) $\forall\ X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$

(ii) $\forall\ X \xrightarrow{f} Y$, a morphism in $\mathcal{C}$; a morphism

$$F(X) \xrightarrow{F(f)} F(Y) \text{ in } \mathcal{D}.$$

such that
- $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)} \quad \forall\ X \in \mathcal{C}.$
- $F(g \underset{\text{in } \mathcal{C}}{\circ} f) = F(g) \underset{\text{in } \mathcal{D}}{\circ} F(f) \quad \forall\ \underset{\text{morphisms in } \mathcal{C}.}{X \xrightarrow{f} Y \xrightarrow{g} Z}$

A <u>contravariant functor</u> $\tilde{F}$ from $\mathcal{C}$ to $\mathcal{D}$ assigns

(i) $X \in \mathcal{C} \rightsquigarrow \tilde{F}(X) \in \mathcal{D}$ (ii) $\underset{\text{in } \mathcal{C}}{X \xrightarrow{f} Y} \rightsquigarrow \tilde{F}(Y) \xrightarrow{\tilde{F}(f)} \tilde{F}(X)$

such that $\tilde{F}(Id_X) = Id_{\tilde{F}(X)}$ & $\tilde{F}(g \circ f) = \tilde{F}(f) \circ \tilde{F}(g)$

Examples. — (a) $\mathcal{C} =$ <u>Groups</u>, $\mathcal{D} =$ <u>Sets</u>

$F : \mathcal{C} \longrightarrow \mathcal{D}$ "forgetful functor".

$F(G) =$ the set G

$G \xrightarrow{f} G' \rightsquigarrow G \xrightarrow{f} G'$

(b) For a topological space X, let $\pi_1(X)$ be the fundamental group of X. More precisely, (as fundamental group depends on a choice of base point), $\pi_1(X, x_0) := \dfrac{\text{gp. of loops based at } x_0}{\text{homotopy relation}}$

$(X, x_0) \longmapsto \pi_1(X, x_0)$

is a (covariant) functor : Pointed Top. Sp. $\longrightarrow$ <u>Groups</u>

category : objects $= (X, x_0)$ where X is a topological space $x_0 \in X$.

morphisms : $\left\{ \begin{array}{l} \text{a continuous maps } X \xrightarrow{f} Y \\ \text{s.t. } f(x_0) = y_0 \end{array} \right\}$

$=: \operatorname{Hom}((X, x_0), (Y, y_0))$

(c) Let $\mathcal{C}$ = category described as follows:

Objects $= \mathbb{N}$
$= \{0, 1, 2, \dots\}$

$\mathrm{Hom}_{\mathcal{C}}(m, n)$ is singleton if $m \leq n$
ϕ otherwise.

Id_n = the unique element of $\mathrm{End}_{\mathcal{C}}(n)$.

composition maps are uniquely given.

Let $\mathcal{D}$ = Category of finite-dimensional vector spaces over K, a field.

$$
\begin{aligned}
F : \mathcal{C} &\longrightarrow \mathcal{D} \\
n &\longmapsto K^n \\
\underset{(m \leq n)}{m \longrightarrow n} &\longmapsto K^m \longrightarrow K^n \\
& \qquad (x_1, \dots, x_m) \longmapsto (x_1, \dots, x_m, 0, \dots, 0)
\end{aligned}
$$

is a covariant functor

(d) <u>Hom functors</u>. Let $\mathcal{C}$ be a category. Let $X \in \mathcal{C}$

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} &\longrightarrow \underline{\mathrm{Sets}} \\
Y &\longmapsto \mathrm{Hom}_{\mathcal{C}}(Y, X) \\
Y \xrightarrow{f} Z &\longmapsto \mathrm{Hom}(Z, X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(f, X)} \mathrm{Hom}(\cancel{Z}Y, X) \\
& \qquad\quad g \longmapsto g \circ f
\end{aligned}
$$

is a contravariant functor.

Analogously $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is a covariant functor.

Remark. — Functors can be composed as usual functions.

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E} \quad \rightsquigarrow \quad \mathcal{C} \xrightarrow{G \circ F} \mathcal{E}$$

<u>Note</u>: covariant $\circ$ covariant = covariant = contra $\circ$ contra

covariant $\circ$ contravariant = contravariant
= contravariant $\circ$ covariant

(1.2) Natural transformations. — Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let F, G be two (covariant) functors. A <u>natural transformation</u> η from F to G assigns $\forall X \in \mathcal{C}$, a morphism

$$F(X) \xrightarrow{\eta_X} G(X)$$

such that for any morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

[If F and G are contravariant, the last diagram is to be changed to

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\uparrow & & \uparrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

.]

We <u>do not have natural transformations</u> between <u>a covariant</u> F & <u>a contravariant</u> G.

If $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ is an isomorphism for every $X \in \mathcal{C}$, we say η is a natural isomorphism between F and G, its inverse being given by $\eta^{-1} : G \to F$

$$\eta^{-1}_X = \text{inverse of } \eta_X.$$

Example of double dual.

Let Vect_K = category of K-vector spaces (K : a field).

$\mathrm{Vect}_K \xrightarrow{\mathbb{D}} \mathrm{Vect}_K$ is a covariant functor

$V \longmapsto V^{**}$ (recall $V^* =$ linear forms $\{V \to K\}$)

The map $V \xrightarrow{\eta_V} V^{**}$

$v \longmapsto \{\xi \in V^* \longmapsto \xi(v) \in K\}$

is then a natural transformation η from $\mathrm{Id}_{\mathrm{Vect}_K}$ (identity functor) and $\mathbb{D}$.

Note. — η_V is not in general an isomorphism. However on the "subcategory" of finite-dimensional vector spaces η is a natural isomorphism $\mathrm{Id}_{\mathrm{Vect}_K} \longrightarrow \mathbb{D}$.

(1.3) Faithful and full functors. — Again let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let F be a (covariant) functor $\mathcal{C} \to \mathcal{D}$.

We say F is <u>faithful</u> if the set map induced by F

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X,Y) &\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ f &\longmapsto F(f) \end{aligned} \qquad \left[\begin{array}{c}\mathrm{Hom}_{\mathcal{D}}(F(Y),F(X)) \\ \text{for contravariant } F\end{array}\right]$$

is one-one. We say F is <u>full</u> if this map is onto.

Example. – (1) <u>Groups</u> $\longrightarrow$ <u>Sets</u> forgetful functor (ca) of page 2 above)

is faithful but not full.

(2) $(X, x_0) \longmapsto \pi_1(X, x_0)$ (Example (b) page 2)

is neither faithful nor full.

(3) By a "concrete" category we mean a category $\mathcal{C}$ which admits a <u>faithful</u> functor $\mathcal{C} \longrightarrow$ <u>Sets</u>

If F is a covariant faithful functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, it allows us to view $\mathcal{C}$ as a <u>subcategory</u> of $\mathcal{D}$.

Subcategory: of a category $\mathcal{A}$, say $\mathcal{A}'$, consists of some objects of $\mathcal{A}$, such that $\mathrm{Hom}_{\mathcal{A}'}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{A}}(X,Y)$ $\forall\, X, Y \in \mathcal{A}'$.

(i.e. $\begin{aligned}\mathcal{A}' &\longrightarrow \mathcal{A} \\ X &\longmapsto X\end{aligned}$ is faithful)

If the inclusion $(\mathcal{A}' \hookrightarrow \mathcal{A})$ is full, we say $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$.

e.g. $\mathrm{Vect}_K^{fd} \subset \mathrm{Vect}_K$ is a full subcategory.

f.d. vector spaces.

(1.4) Equivalence of categories. We say two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if we have two (covariant) functors

$$F : \mathcal{C} \longrightarrow \mathcal{D} \qquad G : \mathcal{D} \longrightarrow \mathcal{C}$$

(in other words F is an equivalence of categories and so is G)

and natural isomorphisms

$$\mathrm{Id}_{\mathcal{C}} \longrightarrow G \circ F$$
$$\mathrm{Id}_{\mathcal{D}} \longrightarrow F \circ G$$

[if F & G are contravariant, we say $\mathcal{C}$ & $\mathcal{D}$ are dual equivalent]

Unraveling the definition. - besides the functors F & G, we need isomorphisms

$$X \xrightarrow{\varphi_X} G(F(X)) \quad \forall X \in \mathcal{C}$$
$$Y \xrightarrow[\psi_Y]{} F(G(Y)) \quad \forall Y \in \mathcal{D}$$

which are natural in X and Y:

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi_{X_1}} & GF(X_1) \\ f \downarrow & & \downarrow GF(f) \\ X_2 & \xrightarrow[\varphi_{X_2}]{} & GF(X_2) \end{array}$$

($X \in \mathcal{C}$, $Y \in \mathcal{D}$) and similar one for Y's commutes.

Isomorphism of categories is a notion stronger than that of equivalence, in that, φ_X and ψ_Y are required to be identity.
$\left(\text{i.e. } G \circ F = \mathrm{Id}_{\mathcal{C}} \text{ and } F \circ G = \mathrm{Id}_{\mathcal{D}} \right)$

Silly example. — $\mathcal{D}$ = a category where $\left|\mathrm{Hom}_{\mathcal{D}}(Y, Y')\right| = 1 \ \forall\, Y, Y' \in \mathcal{D}$.

$\mathcal{C}$ = a category with one object, $*$, and $\mathrm{End}_{\mathcal{C}}(*) = \{\mathrm{Id}_*\}$.

Then $\mathcal{C}$ and $\mathcal{D}$ are equivalent.