Lecture 2

(2.0) **Recall.** For two categories $\mathcal{C}$ and $\mathcal{D}$,

- a functor $F : \mathcal{C} \to \mathcal{D}$ assigns $X \in \mathcal{C} \rightsquigarrow F(X) \in \mathcal{D}$
  [covariant]

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \\ \Psi & & \Psi \\ f & \longmapsto & F(f) \end{array}$$

  s.t. $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ & $F(g \circ f) = F(g) \circ F(f)$

- a natural transformation $\eta : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is given by $\eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ $\forall X \in \mathcal{C}$

  s.t. $\forall f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $\quad G(f) \circ \eta_X = \eta_Y \circ F(f)$

- η is a natural isomorphism if η_X is an iso. $\forall X \in \mathcal{C}$.

- $\mathcal{C}$ and $\mathcal{D}$ are equivalent if we have $\left.\begin{array}{l} F : \mathcal{C} \to \mathcal{D} \\ G : \mathcal{D} \to \mathcal{C} \end{array}\right\}$ functors

$$\left.\begin{array}{l} \varphi : \mathrm{Id}_{\mathcal{C}} \to GF \\ \psi : \mathrm{Id}_{\mathcal{D}} \to FG \end{array}\right\} \text{ natural isomorphisms.}$$

(2.1) **An example.** Let K be a field and define a category

$\underline{\mathrm{Mat}}_K$:

Objects $= \mathbb{N} = \{0, 1, 2, \dots\}$

Morphisms : $\mathrm{Hom}_{\underline{\mathrm{Mat}}_K}(n, m) = \mathcal{M}_{m \times n}(K) \leftarrow$ $m \times n$ matrices w/ entries from K.

Composition $=$ matrix multiplication

$\mathrm{Id}_n = n \times n$ identity matrix.

$\underline{\mathrm{Mat}}_K$ and $\underline{\mathrm{Vect}}_K^{fd}$ (category of finite-dim'l vector spaces over K)

are equivalent.

$$F : \underline{\mathrm{Mat}}_K \longrightarrow \underline{\mathrm{Vect}}_K^{fd}$$
$$n \longmapsto K^n$$

$$\mathrm{Hom}_{\underline{\mathrm{Mat}}_K}(n,m) = \mathcal{M}_{m\times n}(K) = \mathrm{Hom}_{\substack{K\text{-vector}\\ \text{spaces}}}(K^n, K^m)$$

(2.2) Theorem. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence of categories if and only if F is faithful, full and dense. (dense $=$ $\forall\, Y \in \mathcal{D}$, $\exists\, X \in \mathcal{C}$ and an iso. $\beta_Y : Y \xrightarrow{\sim} F(X)$)

[sometimes, also referred to as "essentially surjective"]

Proof. Assume F is an equivalence of categories. That is, we have $G : \mathcal{D} \to \mathcal{C}$, a functor and $\left.\begin{array}{l}\varphi : \mathrm{Id}_{\mathcal{C}} \to GF \\ \psi : \mathrm{Id}_{\mathcal{D}} \to FG\end{array}\right\}$ nat. iso.'s.

$\Rightarrow$ $\psi_Y : Y \xrightarrow{\sim} F(G(Y))$ $\forall\, Y \in \mathcal{D}$. Hence F is dense.

Moreover we have the following sequence of set maps:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "F"] & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \arrow[r, "G"] & \mathrm{Hom}_{\mathcal{C}}(GF(X),GF(Y)) & \varphi_Y \circ f \circ \varphi_X^{-1} \arrow[l, phantom, "\ni"] \\
& & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[u, "\text{bijection}"] & f \arrow[u, mapsto] \arrow[l, phantom, "\ni"]
\end{tikzcd}$$

$\left[\text{recall } \begin{array}{l} \varphi_X : X \xrightarrow{\sim} GF(X) \\ \varphi_Y : Y \xrightarrow{\sim} GF(Y) \end{array} \text{ in } \mathcal{C}\right]$

By definition, $GF(f) = \varphi_Y \circ f \circ \varphi_X^{-1}$

$(\forall\, f : X \to Y$ in $\mathcal{C}$,
$$\begin{tikzcd}
X \arrow[r, "\varphi_X"] \arrow[d, "f"'] & GF(X) \arrow[d, "GF(f)"] \\
Y \arrow[r, "\varphi_Y"'] & GF(Y)
\end{tikzcd}$$
commutes by defn.)

$\Rightarrow$ $\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ is a bijection, i.e.,

F is faithful & full.

Conversely, assume F is faithful, full and dense.

For any $Y \in \mathcal{D}$, choose (and fix) $\overline{Y} \in \mathcal{C}$ and $\beta_Y : Y \xrightarrow{\sim} F(\overline{Y})$

Definition of G. — $Y \in \mathcal{D} \rightsquigarrow G(Y) := \overline{Y} \in \mathcal{C}$

$$\begin{tikzcd}
f \arrow[d, mapsto] & \in & \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2) \arrow[d] \arrow[drr, "\text{define to be } G \text{ on morphisms.}"] & & \\
\beta_{Y_2} \circ f \circ \beta_{Y_1}^{-1} & & \mathrm{Hom}_{\mathcal{D}}(F(\overline{Y}_1), F(\overline{Y}_2)) & & \mathrm{Hom}_{\mathcal{C}}(\overline{Y}_1, \overline{Y}_2) \arrow[ll, "F"', "\text{bijection}"]
\end{tikzcd}$$

Check : G is a functor.

Definition of $\varphi : \mathrm{Id}_{\mathcal{C}} \longrightarrow GF$. – For any $X \in \mathcal{C}$,

$$F(\varphi_X) = \beta_{F(X)} : F(X) \xrightarrow{\sim} F\left(\overline{F(X)}\right).$$

As F is bijective on morphisms

$$\beta_{F(X)} = F\left(\text{a morphism } X \to \overline{F(X)} \text{ in } \mathcal{C}\right)$$

$\uparrow$ define to be φ_X

$\updownarrow$

$$\varphi_X : X \longrightarrow \overline{F(X)} = G(F(X))$$

$F(\varphi_X)$ is an isomorphism $\Rightarrow$ φ_X is an isomorphism. [Homework].
(since F is faithful & full)

Check : φ is a natural isomorphism

[Proof. - Given $f : X_1 \to X_2$ a morphism in $\mathcal{C}$, we need to prove that

$$\begin{tikzcd} X_1 \arrow[r, "\varphi_{X_1}"] \arrow[d, "f"'] & GF(X_1) \arrow[d, "GF(f)"] \\ X_2 \arrow[r, "\varphi_{X_2}"'] & GF(X_2) \end{tikzcd}$$

commutes.

$$G(F(f)) = F^{-1}\left(\beta_{F(X_2)} \cdot F(f) \cdot \beta^{-1}_{F(X_1)}\right) \quad \text{by definition of } G$$

inverse of the bijection $\mathrm{Hom}_{\mathcal{D}}(FA, FB) \xleftarrow{F} \mathrm{Hom}_{\mathcal{C}}(A, B)$

$$= \varphi_{X_2} \circ f \circ \varphi^{-1}_{X_1} \quad \text{as required.} \quad \square \;]$$

Definition of ψ. - $\psi : \mathrm{Id}_{\mathcal{D}} \longrightarrow FG$

$$\psi_Y = \beta_Y : Y \xrightarrow{\sim} F(\overline{Y}) = FG(Y)$$

Check (easy) : Ψ is a natural iso.

Hence $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories. $\square$

(2.3) Remark. – Though we stated and proved the theorem above for covariant functors, similar assertion with verbatim proof works when both F & G are contravariant.

More conceptually, given a category $\mathcal{C}$, one can (artificially) define, category dual / opposite to $\mathcal{C}$, denoted by $\mathcal{C}^{op}$ as:

$$\text{Objects of } \mathcal{C}^{op} := \text{Objects of } \mathcal{C}$$

$$\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(Y,X) \quad \left[\begin{array}{l}\text{Notation } f \in \mathrm{Hom}_{\mathcal{C}}(Y,X) \\ \leftrightarrow \tilde{f} \in \mathrm{Hom}_{\mathcal{C}^{op}}(X,Y)\end{array}\right]$$

Id_X in $\mathcal{C}^{op}$ $:= \mathrm{Id}_X$ in $\mathcal{C}$ and composition in $\mathcal{C}^{op}$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}^{op}}(Y,Z) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}^{op}}(X,Z) \\
(\tilde{g}, \tilde{f}) & \longmapsto & \tilde{f} \circ \tilde{g} := \widetilde{g \circ f} \\
\| & & \| \\
\mathrm{Hom}_{\mathcal{C}}(Y,X) \times \mathrm{Hom}_{\mathcal{C}}(Z,Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Z,X) \\
(g, f) & \longmapsto & g \circ f
\end{array}$$

With this definition in mind, Contravariant functors $\mathcal{C} \to \mathcal{D}$ are the same as covariant functors $\mathcal{C}^{op} \longrightarrow \mathcal{D}$.

(2.4) Product of categories and multi-functors. —

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories. Define $\mathcal{C}_1 \times \mathcal{C}_2$ as:

Objects of $\mathcal{C}_1 \times \mathcal{C}_2$ are (X_1, X_2) where $X_j \in \mathcal{C}_j$ $(j=1,2)$.

$$\mathrm{Hom}_{\mathcal{C}_1 \times \mathcal{C}_2}\big((X_1, X_2), (Y_1, Y_2)\big) := \mathrm{Hom}_{\mathcal{C}_1}(X_1, Y_1) \times \mathrm{Hom}_{\mathcal{C}_2}(X_2, Y_2)$$

Composition in $\mathcal{C}_1 \times \mathcal{C}_2$ is componentwise

$$\mathrm{Id}_{(X_1, X_2)} = (\mathrm{Id}_{X_1}, \mathrm{Id}_{X_2}).$$

This construction will allow us to speak of functors in "several variables", covariant in some, contravariant in some other variables.

(2.5) Example. Let $\mathcal{C} = \mathrm{Vect}_K$ (category of K-vector spaces, where K is a field).

$$\left.\begin{aligned} F : \mathcal{C} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (V, W) &\longmapsto V^* \otimes W \end{aligned}\right\} \begin{aligned}&\text{Contra in } 1^{st} \\ &\text{covariant in } 2^{nd}\end{aligned}$$

$$G : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

$$(V, W) \longmapsto \mathrm{Hom}_K(V, W) \quad \text{(again a vector space)}$$

Recall, there is a linear map

$$\varphi_{V,W} : V^* \otimes W \longrightarrow \mathrm{Hom}_K(V, W)$$

$$\xi \otimes w \longmapsto \{ v \mapsto \xi(v) \cdot w \}$$

[see, for example, Lecture 14, page 7 of Algebra I]

φ is a natural transformation from F to G. It is a (nat.) isomorphism, assuming V is finite-dim'l:

$$\mathcal{C}' = \mathrm{Vect}_K^{fd} \hookrightarrow \mathrm{Vect}_K = \mathcal{C}$$

$$\mathcal{C}' \times \mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C} \quad \text{are naturally isomorphic via } \varphi$$