Lecture 3

(3.0) Errata. Recall that we say that two categories $\mathcal{C}$ & $\mathcal{D}$ are equivalent if we are given two functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and two natural isomorphisms $\varphi : Id_{\mathcal{C}} \xrightarrow{\sim} GF$ and $\psi : Id_{\mathcal{D}} \xrightarrow{\sim} FG$.

Last time (Theorem 2.2 page 2) we stated that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories (i.e. $\exists\, G : \mathcal{D} \to \mathcal{C}$ and φ & ψ as above) if and only if F is faithful, full and dense (or essentially surjective). In the proof of: F is equivalence $\Rightarrow$ F is faithful and full; on page 3 of Lecture 2, the argument only shows that F is faithful (i.e., $\forall\, X_1, X_2 \in \mathcal{C}$, the set map $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2))$ is one-one). To prove surjectivity (i.e., to prove that F is full) we need the following diagram:

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X_1),F(X_2)) \arrow[r, "G"] \arrow[ddr] & \mathrm{Hom}_{\mathcal{C}}(GF(X_1), GF(X_2)) \arrow[d, "\varphi_{X_2}^{-1}\circ - \circ \varphi_{X_1}"] \\
g \arrow[d, mapsto] & \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \arrow[d, "F"] \\
F(\varphi_{X_2})^{-1} \circ FG(g) \circ F(\varphi_{X_1}) & \mathrm{Hom}_{\mathcal{C}}(F(X_1), F(X_2))
\end{tikzcd}
$$

conclude that it is a bijection since $FG(g) = \psi_{F(X_2)} \circ g \circ \psi_{F(X_1)}^{-1}$ and hence F is onto.

(3.1) Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and assume that we have two functors $\mathcal{C} \underset{F_2}{\overset{F_1}{\rightleftarrows}} \mathcal{D}$. We say (F_1, F_2) form an <u>adjoint pair</u> (or F_1 is <u>a left adjoint</u> of F_2; or F_2 is <u>a right adjoint</u> of F_1) if we have natural isomorphisms $\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y)$ $\forall\, X \in \mathcal{C},\ Y \in \mathcal{D}$.

Sometimes people also write: (F_1, F_2, β) is an adjunction.

<u>What is β?</u> β is a natural isomorphism between two functors $\mathcal{C} \times \mathcal{D} \longrightarrow \underline{\mathrm{Sets}}$. Namely, $\mathcal{C} \times \mathcal{D} \xrightarrow[\mathrm{Id}_{\mathcal{C}} \times F_2]{} \mathcal{C} \times \mathcal{C} \xrightarrow[\mathrm{Hom}_{\mathcal{C}}(-,-)]{} \underline{\mathrm{Sets}}$ and $\mathcal{C} \times \mathcal{D} \xrightarrow[F_1 \times \mathrm{Id}_{\mathcal{D}}]{} \mathcal{D} \times \mathcal{D} \xrightarrow[\mathrm{Hom}_{\mathcal{D}}(-,-)]{} \underline{\mathrm{Sets}}$. As a diagram, we remember this as:

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{D} \arrow[r, "\mathrm{Id}_{\mathcal{C}} \times F_2"] \arrow[d, "F_1 \times \mathrm{Id}_{\mathcal{D}}"'] \arrow[dr, phantom, "\overset{\text{nat.}}{\Swarrow \beta : \text{iso}}"] & \mathcal{C} \times \mathcal{C} \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(-,-)"] \\
\mathcal{D} \times \mathcal{D} \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(-,-)"'] & \underline{\mathrm{Sets}}
\end{tikzcd}$$

Note. – These functors are "mixed", i.e., contravariant in 1^{st} argument and covariant in the 2^{nd}!

More explicitly, β is the data of a <u>bijection</u>

$$\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y) \quad \forall \begin{matrix} X \in \mathcal{C} \\ Y \in \mathcal{D} \end{matrix}$$

such that for every pair of morphisms $\begin{matrix} X \xrightarrow{f} X' \text{ in } \mathcal{C} \\ Y \xleftarrow[g]{} Y' \text{ in } \mathcal{D} \end{matrix}$

the following diagram commutes:

$$\begin{tikzcd}
F_2(g)\circ\alpha\circ f & \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y)) \arrow[r, "\beta_{X,Y}"] & \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y) \ni g\circ\gamma\circ F_1(f) \\
\alpha \in \arrow[u, mapsto] & \mathrm{Hom}_{\mathcal{C}}(X', F_2(Y')) \arrow[u] \arrow[r, "\beta_{X',Y'}"'] & \mathrm{Hom}_{\mathcal{D}}(F_1(X'), Y') \ni \gamma \arrow[u] \arrow[u, mapsto, shift left=12ex]
\end{tikzcd}$$

(3.2) An example. — Let G be a group and let G-<u>Sets</u> be the category of sets together with a G-action (see Problem Set 1)

We have a forgetful functor G-<u>Sets</u> $\longrightarrow$ <u>Sets</u> and

<u>Sets</u> $\longrightarrow$ G-<u>Sets</u>

$Y \longmapsto G\times Y$ with G-action on the first component

$Y_1 \xrightarrow{f} Y_2 \longmapsto G\times Y_1 \xrightarrow[(\mathrm{Id}_G\times f)]{} G\times Y_2$.

<u>Lemma</u>. There is a natural bijection $\forall \begin{matrix} X\in \underline{\text{Sets}} \\ Y \in G\text{-}\underline{\text{Sets}} \end{matrix}$

$$\mathrm{Hom}_{\underline{\text{Sets}}}(X, Y) \longrightarrow \mathrm{Hom}_{G\text{-}\underline{\text{Sets}}}(G\times X, Y)$$

Thus, $\underline{\text{Sets}} \underset{F_2 = \text{forgetful}}{\overset{F_1}{\rightleftarrows}} G\text{-}\underline{\text{Sets}}$ $\quad \left(F_1(X) = G \times X \right)$

is a pair (F_1, F_2) of adjoint functors.

Proof of the lemma. — Let X be a set and Y be a G-set.

Define
$$\beta_{X,Y} : \operatorname{Hom}_{\underline{\text{Set}}}(X, Y) \longrightarrow \operatorname{Hom}_{G\text{-}\underline{\text{Sets}}}(\underset{\underset{G\times X}{\|}}{F_1(X)}, Y)$$
$$f \longmapsto \{ (g, x) \mapsto g \cdot f(x) \}$$

$\underline{\text{Check}}$. $\beta_{X,Y}$ is a bijection with inverse given by:
$$\operatorname{Hom}_{G\text{-}\underline{\text{Sets}}}(G \times X, Y) \longrightarrow \operatorname{Hom}_{\underline{\text{Sets}}}(X, Y)$$
$$g \longmapsto \{ x \mapsto g(\underbrace{e}_{\text{identity of } G}, x) \}$$

Naturality. — Let $X \xrightarrow{f} X'$ be a set map

$Y \xleftarrow{g} Y'$ be a map of G-sets.

$$\begin{tikzcd}
\operatorname{Hom}_{\underline{\text{Sets}}}(X, Y) \arrow[r] & \operatorname{Hom}_{G\text{-}\underline{\text{Sets}}}(G \times X, Y) \\
\operatorname{Hom}_{\underline{\text{sets}}}(X', Y') \arrow[r] \arrow[u] & \operatorname{Hom}_{G\text{-}\underline{\text{Sets}}}(G \times X', Y') \arrow[u]
\end{tikzcd}$$

$$g \circ \alpha \circ f \longmapsto \mathrm{II} \in \operatorname{Hom}_{G\text{-}\underline{\text{Sets}}}(G \times X, Y) \ni \mathrm{I}$$
$$\alpha \longmapsto g \circ \alpha \circ f, \qquad \alpha \in \operatorname{Hom}_{\underline{\text{sets}}}(X', Y')$$
$$\alpha \longmapsto \{ (\sigma, x') \mapsto \sigma \cdot \alpha(x') \} \longmapsto \mathrm{I}$$

I : $\cancel{g\,x}$ $(\sigma, x) \longmapsto g(\sigma \cdot (\alpha(f(x))))$

II : $(\sigma, x) \longmapsto \sigma \cdot (g(\alpha(f(x))))$

equal since g is a map of G-Sets $\square$

(3.3) The example above can be generalized as follows: for every group G and a subgroup H of G; we have

$$\underline{H\text{-Sets}} \underset{\substack{F_2 = \text{restriction} \\ \text{of action}}}{\overset{F_1}{\rightleftarrows}} \underline{G\text{-Sets}}$$

For $X \in \underline{H\text{-Sets}}$, $F_1(X) = G \underset{H}{\times} X := \dfrac{G \times X}{(\sigma_1\sigma_2, x) \sim (\sigma_1, \sigma_2 x)}$
"induction" $\qquad \forall\, \sigma_1 \in G,\ \sigma_2 \in H$

(If $H = \{e\}$, $\underline{H\text{-Sets}} = \underline{\text{Sets}}$ and we recover §3.2).

(3.4) Unit and counit of adjunction. — Again let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $\mathcal{C} \underset{F_2}{\overset{F_1}{\rightleftarrows}} \mathcal{D}$ two functors, together with natural iso.

$$\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y)$$

[check this]

(1) We get a <u>natural transformation</u> $F_1F_2 \xrightarrow{\varepsilon} Id_{\mathcal{D}}$

$$\varepsilon_Y = \beta_{F_2(Y), Y}\left(Id_{F_2(Y)}\right) \in \mathrm{Hom}_{\mathcal{D}}(F_1F_2(Y), Y)$$

called "counit of adjunction".

(2) Similarly, $\beta_{X, F_1(X)} : \mathrm{Hom}_{\mathcal{C}}(X, F_2F_1(X)) \to \mathrm{Hom}_{\mathcal{D}}(F_1(X), F_1(X))$

Define $\eta_X = \beta^{-1}_{X, F_1(X)}(\mathrm{Id}_{F_1(X)}) : X \longrightarrow F_2F_1(X)$

Thus we get a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \to F_2F_1$ called "unit of adjunction".

Lemma. – The following two compositions are identity

$$\left.\begin{array}{l} F_1 \longrightarrow F_1F_2F_1 \longrightarrow F_1 \\ F_2 \longrightarrow F_2F_1F_2 \longrightarrow F_2 \end{array}\right\} \text{arrows are natural trans.}$$

Proof. – Let us spell out the first sequence of natural transformations. For $C \in \mathcal{C}$, it is

$$F_1(C) \xrightarrow{F_1(\eta_C)} F_1(F_2F_1(C)) \xrightarrow{\varepsilon_{F_1(C)}} F_1(C)$$

We need to prove : $\varepsilon_{F_1(C)} \circ F_1(\eta_C) = \mathrm{Id}_{F_1(C)}$

In the commutative diagram of page 3 above, take

$X = C$, $X' = F_2F_1(C)$, $f = \eta_C$, $Y = Y' = F_1(C)$

$g = \mathrm{Id}_Y$.

The diagram becomes

$$\begin{tikzcd}
\alpha \circ \eta_C \in \operatorname{Hom}_{\mathcal{C}}(C, F_2F_1(C)) \arrow[r, "\beta_{C,F_1(C)}"] & \operatorname{Hom}_{\mathcal{D}}(F_1(C), F_1(C)) \ni \gamma \circ F_1(\eta_C) \\
\alpha \in \operatorname{Hom}_{\mathcal{C}}(F_2F_1(C), F_2F_1(C)) \arrow[u] \arrow[r, "\beta_{F_2F_1(C), F_1(C)}"'] & \operatorname{Hom}_{\mathcal{D}}(F_1F_2F_1(C), F_1(C)) \ni \gamma \arrow[u]
\end{tikzcd}$$

Apply the arrows to $\alpha = \mathrm{Id}_{F_2F_1(C)}$ to get

$$\underbrace{\beta_{C,F_1(C)}(\eta_C)}_{\substack{\| \\ \mathrm{Id}_{F_1(C)} \\ (\text{by defn. of } \eta_C)}} = \underbrace{\beta_{F_2F_1(C),F_1(C)}(\mathrm{Id}_{F_2F_1(C)})}_{\substack{\| \\ \varepsilon_{F_1(C)} \\ (\text{by defn. of } \varepsilon_{F_1(C)})}} \circ F_1(\eta_C)$$

This finishes the proof for the first sequence. The second one is proved analogously. $\square$