(4.0) Recall: for two categories $\mathcal{C}$ and $\mathcal{D}$, and a pair of functors

$\mathcal{C} \underset{F_2}{\overset{F_1}{\rightleftarrows}} \mathcal{D}$, we say (F_1, F_2) is an adjoint pair (see §3.1 p.2)

if we are given an isomorphism of functors

$$\begin{tikzcd} & \mathcal{C}\times\mathcal{C} \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}(-,-)"] \arrow[dd, phantom, "\Downarrow \beta"] & \\ \mathcal{C}\times\mathcal{D} \arrow[ur, "\mathrm{Id}_{\mathcal{C}}\times F_2"] \arrow[dr, "F_1\times \mathrm{Id}_{\mathcal{D}}"'] & & \underline{\mathrm{Sets}} \\ & \mathcal{D}\times\mathcal{D} \arrow[ur, "\mathrm{Hom}_{\mathcal{D}}(-,-)"'] & \end{tikzcd}$$

That is, there are bijections $\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, F_2Y) \to \mathrm{Hom}_{\mathcal{D}}(F_1X, Y)$

$\forall\, X \in \mathcal{C},\ Y \in \mathcal{D}$; natural in X & Y.

We defined $\varepsilon : F_1F_2 \to \mathrm{Id}_{\mathcal{D}}$ natural transformations

$\eta : \mathrm{Id}_{\mathcal{C}} \to F_2F_1$

by $\varepsilon_Y := \beta_{F_2Y,Y}(\mathrm{Id}_{F_2Y}) : F_1F_2Y \to Y \quad \forall\, Y \in \mathcal{D}$

$\eta_X := \beta^{-1}_{X,F_1X}(\mathrm{Id}_{F_1X}) : X \to F_2F_1X \quad \forall\, X \in \mathcal{C}$

and proved that the resulting sequences of natural transformations

$F_1 \to F_1F_2F_1 \to F_1$

$F_2 \to F_2F_1F_2 \to F_2$

compose to give identity. (Lemma 3.4 page 6)

i.e. $\forall\, X \in \mathcal{C}$ $\quad F_1(X) \xrightarrow{F_1(\eta_X)} F_1F_2F_1(X) \xrightarrow{\varepsilon_{F_1(X)}} F_1(X) \quad = \mathrm{Id}_{F_1(X)}$

$\forall\, Y \in \mathcal{C}$ $\quad F_2(Y) \xrightarrow[\eta_{F_2(Y)}]{} F_2F_1F_2(Y) \xrightarrow[F_2(\varepsilon_Y)]{} F_2(Y) \quad = \mathrm{Id}_{F_2(Y)}$

(4.1) Now we state and prove the converse.

Again consider a pair of functors $\mathcal{C} \underset{F_2}{\overset{F_1}{\rightleftarrows}} \mathcal{D}$. Assume that we are given

Natural transformations $\varepsilon : F_1 F_2 \to \mathrm{Id}_{\mathcal{D}}$

$\eta : \mathrm{Id}_{\mathcal{C}} \to F_2 F_1$

s.t. the resulting compositions are identity:

$$F_1 \to F_1 F_2 F_1 \to F_1 \quad \text{and} \quad F_2 \to F_2 F_1 F_2 \to F_2$$

Prop. With the hypotheses listed above, (F_1, F_2) are adjoint with $\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, F_2 Y) \to \mathrm{Hom}_{\mathcal{D}}(F_1 X, Y)$ given by

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, F_2 Y) \arrow[rr, "\beta_{X,Y} := (\varepsilon_Y \circ -) \circ F_1 = \varepsilon_Y \circ F_1(-)"] \arrow[d, "F_1"] & & \mathrm{Hom}_{\mathcal{D}}(F_1 X, Y) \\
\mathrm{Hom}_{\mathcal{D}}(F_1 X, F_1 F_2 Y) \arrow[urr, "\varepsilon_Y \circ -"'] & &
\end{tikzcd}$$

(i.e. $\beta_{X,Y}(f) = \varepsilon_Y \circ F_1(f)$)

Proof. As a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(-,-) \circ (\mathrm{Id}_{\mathcal{C}} \times F_2)$ to $\mathrm{Hom}_{\mathcal{D}}(-,-) \circ (F_1 \times \mathrm{Id}_{\mathcal{D}})$ (see the picture on the previous page), we can describe β as the following composition of natural transformations (thus proving naturality in X & Y)

$$
\begin{tikzcd}
 & \mathcal{C}\times\mathcal{C} \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}(-,-)"] \arrow[d, phantom, "\Downarrow (F_1\times F_1)"] & \\
\mathcal{C}\times\mathcal{D} \arrow[ur, "\mathrm{Id}_{\mathcal{C}}\times F_2"] \arrow[r, "F_1\times F_1F_2"] \arrow[dr, "F_1\times \mathrm{Id}_{\mathcal{D}}"'] & \mathcal{D}\times\mathcal{D} \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(-,-)"] \arrow[dl, phantom, near start, "\swarrow \mathrm{Id}_{F_1}\times\varepsilon"] \arrow[dr, phantom, near start, "\Downarrow \mathrm{id}"] & \underline{\mathrm{Sets}} \\
 & \mathcal{D}\times\mathcal{D} \arrow[ur, "\mathrm{Hom}_{\mathcal{D}}(-,-)"'] &
\end{tikzcd}
$$

Here we view $(F_1\times F_1)$ as a natural transformation between

$$\mathrm{Hom}_{\mathcal{C}}(-,-)\circ(\mathrm{Id}_{\mathcal{C}}\times F_2) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(-,-)\circ(F_1\times F_1F_2)$$

<u>$\beta_{X,Y}$ is a bijection</u> : The following pictures completes the triangle in the statement of the proposition.

$$
\begin{tikzcd}
 & & \mathrm{Hom}_{\mathcal{C}}(F_2F_1X, F_2(Y)) \arrow[dll, "-\circ\eta_X"'] \\
\mathrm{Hom}_{\mathcal{C}}(X, F_2Y) \arrow[rr, shift right, "\beta_{X,Y}"'] \arrow[d, "F_1"'] & & \mathrm{Hom}_{\mathcal{D}}(F_1X, Y) \arrow[ll, shift right, "\gamma_{X,Y}"'] \arrow[u, "F_2"'] \\
\mathrm{Hom}_{\mathcal{D}}(F_1X, F_1F_2Y) \arrow[urr, "\varepsilon_Y\circ -"'] & &
\end{tikzcd}
$$

We claim that $\gamma_{X,Y}$ is the inverse of $\beta_{X,Y}$. $\forall f\in \mathrm{Hom}_{\mathcal{C}}(X, F_2Y)$

$$\gamma_{X,Y}\circ\beta_{X,Y}(f) = F_2(\varepsilon_Y)\circ \boxed{F_2F_1(f)\circ\eta_X} \xrightarrow[\text{of } \eta]{\text{by naturality}} \eta_{F_2Y}\circ f$$

$$= \boxed{F_2(\varepsilon_Y) \circ \eta_{F_2(Y)}} \circ f$$

$\longrightarrow \; = \mathrm{Id}_{F_2 Y}$ since

$= f$ $\qquad \mathrm{Id}_{F_2 Y} = \{ F_2 Y \xrightarrow{\eta_{F_2 Y}} F_2 F_1 F_2 Y \xrightarrow{F_2(\varepsilon_Y)} F_2 Y \}$

Similarly $\beta_{X,Y} \circ \gamma_{X,Y}(g) = \boxed{\varepsilon_Y \circ F_1 F_2(g)} \circ F_1(\eta_X)$

$\forall g \in \mathrm{Hom}_{\mathcal{D}}(F_1 X, Y)$ $\qquad = g \circ \boxed{\varepsilon_{F_1(X)} \circ F_1(\eta_X)} = g$

$\square$

(4.2) *Yoneda's Lemma.* Let $\mathcal{C}$ be a category. For any $X \in \mathcal{C}$, let us abbreviate $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \longrightarrow \underline{\mathrm{Sets}}$ contravariant

and $h^X : \mathrm{Hom}_{\mathcal{C}}(X, -) \longrightarrow \underline{\mathrm{Sets}}$.
covariant

For definiteness, let us focus on contravariant case.

Theorem. Let $\mathrm{Func}(\mathcal{C}^{op}, \underline{\mathrm{Sets}})$ be the category of contra. functors $\mathcal{C} \to \underline{\mathrm{Sets}}$. Then $X \longmapsto h_X$ is a faithful and full functor $\mathcal{C} \xrightarrow{h_\cdot} \mathrm{Func}(\mathcal{C}^{op}, \underline{\mathrm{Sets}})$.
(covariant)

$h_\bullet$ is usually called <u>Yoneda embedding</u>.

Proof.– Step 1. $h_\bullet$ is a functor. Let $f : X \to X'$ be a morphism in $\mathcal{C}$. $h_f : h_X \longrightarrow h_{X'}$ is the following natural transformation:

$$\begin{array}{cccc} h_{f;Y} : & h_X(Y) & \longrightarrow & h_{X'}(Y) \\ & \| & & \| \\ & \mathrm{Hom}(Y,X) & \xrightarrow[f\circ -]{} & \mathrm{Hom}(Y,X') \end{array}$$

We only need to show that for any $Y \xrightarrow{g} Y'$, the following diagram commutes, to prove that h_f is a nat. trans.

$$\begin{tikzcd} h_X(Y') \arrow[r, "h_X(g)"] \arrow[d, "h_{f;Y'}"'] & h_X(Y) \arrow[d, "h_{f;Y}"] \\ h_{X'}(Y') \arrow[r, "h_{X'}(g)"'] & h_{X'}(Y) \end{tikzcd}$$

(contravariance of h_X)

[recall: $h_X(g) = \_ \circ g$]

i.e. $\forall\ \alpha \in h_X(Y') = \mathrm{Hom}_{\mathcal{C}}(Y', X)$; $(f \circ \alpha) \circ g = f \circ (\alpha \circ g)$

which is true by associativity of composition.

Thus $f \in \mathrm{Hom}_{\mathcal{C}}(X, X') \rightsquigarrow h_f \in \mathrm{Hom}_{\mathrm{Func}(\mathcal{C}^{op}, \underline{\mathrm{Sets}})}(h_X, h_{X'})$

Finally to prove that $\begin{array}{l} X \rightsquigarrow h_X \\ f \rightsquigarrow h_f \end{array}$ is a functor, We need to check: $h_{Id_X} = Id_{h_X}$ (true since $h_f = f \circ -$, see definition above). ; and $h_{f \circ f'} = h_f \circ h_{f'}$ (again true because of associativity of composition).

Step 2. $h_\bullet : \mathcal{C} \longrightarrow \mathrm{Func}(\mathcal{C}^{op}, \underline{\mathrm{Sets}})$ is faithful.

That is,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, X') & \longrightarrow & \mathrm{Hom}_{\mathrm{Func}(\mathcal{C}^{op}, \underline{\mathrm{Sets}})}(h_X, h_{X'}) \\ \Psi & & \Psi \\ f & \longmapsto & h_f \end{array} \quad - (*)$$

is one to one.

Assume $h_f = h_{f'}$. This means $h_{f;Y} = h_{f';Y}$ $\forall Y \in \mathcal{C}$.

Set $Y = X$; $\quad h_{f;X} = h_{f';X} : \mathrm{Hom}(X, X) \longrightarrow \mathrm{Hom}(X, X')$

$$\begin{array}{ccc} \Psi & & \\ Id_X & \longmapsto & f \\ & \longmapsto & f' \end{array} \checkmark$$

$$\Rightarrow f = h_{f;X}(Id_X) = h_{f';X}(Id_X) = f'.$$

Step 3. $h_\bullet$ is full. i.e., the set map $f \mapsto h_f$ in $(*)$ is surjective.

Let $\eta : h_X \longrightarrow h_{X'}$ be a natural transformation.

We want to prove that $\eta = h_f$ for some $f : X \to X'$ in $\mathcal{C}$

Again, consider $\eta_X : h_X(X) \longrightarrow h_{X'}(X)$

$$\begin{array}{ccc} h_X(X) & \longrightarrow & h_{X'}(X) \\ \| & & \| \\ \mathrm{Hom}_{\mathcal{C}}(X,X) & & \mathrm{Hom}_{\mathcal{C}}(X,X') \end{array}$$

Define $f := \eta_X(\mathrm{Id}_X)$. We claim that $\forall\, Y \in \mathcal{C}$,

$\eta_Y = f \circ -$; i.e. $\forall\, g \in h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$

$$\eta_Y(g) = f \circ g \in \mathrm{Hom}_{\mathcal{C}}(Y, X') = h_{X'}(Y).$$

Since η is a natural transformation, we get a commutative diagram:

$$\begin{tikzcd} \mathrm{Id}_X \in h_X(X) \arrow[r, "\eta_X"] \arrow[d, "h_X(g)"'] & h_{X'}(X) \arrow[d, "h_{X'}(g)\ (= - \circ g)"] \\ h_X(Y) \arrow[r, "\eta_Y"'] & h_{X'}(Y) \end{tikzcd}$$

$$\Rightarrow \left(h_{X'}(g) \circ \eta_X\right)(\mathrm{Id}_X) = \left(\eta_Y \circ h_X(g)\right)(\mathrm{Id}_X)$$

i.e., $f \circ g = \eta_Y(g)$ as required $\square$

(4.3) Definition. — Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \underline{\text{Sets}}$ be a covariant (resp. contravariant) functor. We say F is <u>representable</u> if $\exists\, X \in \mathcal{C}$ and a natural isomorphism

$$h^X = \mathrm{Hom}_{\mathcal{C}}(X, -) \xrightarrow{\sim} F \quad (\text{resp. } h_X \xrightarrow{\sim} F)$$

Remark. — "Universal problems" in mathematics can be conveniently phrased as "is this functor representable?"

Lemma. — Assume F is a contravariant functor (for definiteness) from $\mathcal{C}$ to $\underline{\text{Sets}}$. If F is representable and $X \in \mathcal{C}$ is ~~the~~ an object representing F, then X is unique up to unique iso.

Proof. — Let X and X' be two objects representing F, i.e., F is naturally iso to h_X and $h_{X'}$. Then we are given

$$\xi_X : h_X \xrightarrow{\sim} F \qquad \Rightarrow \xi_{X'}^{-1} \circ \xi_X : h_X \xrightarrow{\sim} h_{X'}.$$
$$\xi_{X'} : h_{X'} \xrightarrow{\sim} F$$

By Thm. (4.2) this iso. comes from a unique $f : X \to X'$ isomorphism and the lemma is proved. $\square$

(4.4) An example. Let $\mathcal{C} = \mathrm{Vect}_K$ (category of vector spaces over a fixed field K). Let $V, W \in \mathcal{C}$.

Define $\mathcal{C} \xrightarrow{T_{V,W}} \underline{\text{Sets}}$ (actually $\underline{\text{Sets}}$ can be replaced by Vect_K again)

$$X \longmapsto \{\text{bilinear maps } V \times W \to X\} =: T_{V,W}(X)$$

$$X \xrightarrow{f} X' \longmapsto T_{V,W}(X) \xrightarrow[f \circ -]{} T_{V,W}(X')$$

$T_{V,W}$ is representable by $V \underset{K}{\otimes} W$, i.e. $\eta : \text{Hom}(V \otimes W, -) \xrightarrow{\sim} T_{V,W}$ natural iso, namely

(i) $\exists$ a bilinear map $V \times W \longrightarrow V \underset{K}{\otimes} W$

$$b_{V,W} := \eta_{V \underset{K}{\otimes} W}(\text{Id}_{V \underset{K}{\otimes} W}) \in T_{V,W}(V \underset{K}{\otimes} W)$$

(ii) $\forall$ bilinear map $V \times W \xrightarrow{a} X$, $\exists !\ V \underset{K}{\otimes} W \xrightarrow{\tilde{a}} X$ in $\mathcal{C}$

s.t.

$$\begin{tikzcd} V \times W \arrow[r, "b_{V,W}"] \arrow[d, "a"'] & V \otimes W \arrow[dl, "\tilde{a}"] \\ X & \end{tikzcd}$$

$$\left[\text{use. } \eta_X : \text{Hom}_{\mathcal{C}}(V \otimes W, X) \longrightarrow T_{V,W}(X)\right.$$
$$\tilde{a} \in \text{Hom}_{\mathcal{C}}(V \otimes W, X) \longleftrightarrow a \in T_{V,W}(X)$$

Commutativity of the diagram above follows from naturality of η – same argument as on page 7]