Lecture 5

(5.0) Recall last time we studied the Yoneda embedding. For a category $\mathcal{C}$, we have a faithful and full functor (covariant) $h_{\bullet} : \mathcal{C} \longrightarrow \mathrm{Func}(\mathcal{C}^{op}, \underline{\mathrm{Sets}})$ (category of contra functors $\mathcal{C} \to \underline{\mathrm{Sets}}$)

Homework $\forall F \in \mathrm{Func}(\mathcal{C}^{op}, \underline{\mathrm{Sets}})$ and $X \in \mathcal{C}$

$$\mathrm{Hom}_{\mathrm{Func}(\mathcal{C}^{op}, \underline{\mathrm{Sets}})}(h_X, F) = F(X)$$

For a covariant (resp. contravariant) functor $F : \mathcal{C} \to \underline{\mathrm{Sets}}$ we say F is representable, by $X \in \mathcal{C}$, if we have an ~~iso~~ natural isomorphism of functors $F \xrightarrow{\sim} h^X = \mathrm{Hom}_{\mathcal{C}}(X, -)$ (resp. $F \xrightarrow{\sim} h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$).

(5.1) Some examples of representable functors from topology.

(a) $\pi_1(X, x_0)$. Let $\underline{\mathrm{HTop}}$ be a category defined as:

Objects : pair (X, x_0) where X is a topological space and $x_0 \in X$.

Morphisms : $\dfrac{\text{continuous maps } f : X \to Y \text{ s.t. } f(x_0) = y_0}{\text{base point preserving homotopies (equivalence rel}^{n})}$

$S^1 := \{ z \in \mathbb{C} \text{ s.t. } |z| = 1 \}$ with base point 1 is an object of $\underline{\mathrm{HTop}}$.

$$\underline{\mathrm{HTop}} \longrightarrow \underline{\mathrm{Gps}}$$
$$(X, x_0) \longmapsto \pi_1(X, x_0) = \mathrm{Hom}_{\mathrm{HTop}}((S^1, 1), (X, x_0))$$

(b) $\underline{\mathrm{HTop}}^{pc}$ = ~~sub~~category of paracompact topological spaces (morphisms = continuous maps upto homotopy)

Fix $n \geq 1$ and let $\mathrm{Vect}_n : \underline{\mathrm{HTop}}^{pc} \longrightarrow \underline{\mathrm{Sets}}$ be contravariant functor defined by

$$\mathrm{Vect}_n(X) = \{ \text{rank } n \text{ vector bundles on } X \} / \text{isomorphism}$$

$X \xrightarrow{f} Y \rightsquigarrow \mathrm{Vect}_n(Y) \xrightarrow{f^*} \mathrm{Vect}_n(X)$ "pull-back of vector bundles"

Theorem (see Milnor Stasheff : Characteristic classes) (Thm 5.6 page 65)

$\exists$ $Gr(n, \infty) \in$ ~~Vect~~ $\underline{\mathrm{HTop}}^{pc}$ s.t.

$$\mathrm{Vect}_n(X) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathrm{HTop}}^{pc}}(X, Gr(n, \infty))$$

(called "classifying space of GL_n")

(5.2) Last time we advertised the slogan

"universal properties $\subset$ representability of functors"

This idea is heavily used to define mathematical objects (e.g. Hilbert schemes, Stacks, Moduli spaces ...).

Let us define direct sums and direct products this way.

So, let $\mathcal{C}$ be a category and I be a set. Let $X_i \in \mathcal{C}$ be given, $\forall\, i \in I$. Below, we write $\mathfrak{X} = \{X_i\}_{i \in I}$ the set of objects that is given to us. Define

$$\begin{aligned} h^{\mathfrak{X}} : \mathcal{C} &\longrightarrow \underline{\mathrm{Sets}} \\ Y &\longmapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y) \end{aligned}$$

$$Y \xrightarrow{f} Y' \rightsquigarrow \prod_{i \in I} \mathrm{Hom}(X_i, Y) \longrightarrow \prod_{i \in I} \mathrm{Hom}(X_i, Y')$$

$$(g_i) \longmapsto (f \circ g_i)$$

Similarly $h_{\mathfrak{X}} : \mathcal{C} \longrightarrow \underline{\mathrm{Sets}}$

$$Y \longmapsto \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$$

$h^{\mathfrak{X}}$ and $h_{\mathfrak{X}}$ are, respectively, covariant & contravariant functors.

Definition. If $h^{\mathcal{X}}$ is representable, an object representing it is called <u>the direct sum</u> of $\{X_i\}_{i\in I}$, denoted by $\bigoplus_{i\in I} X_i$ (sometimes also by $\coprod_{i\in I} X_i$).

Similarly, if $h_{\mathcal{X}}$ is representable, an object representing it is called <u>the direct product</u> of $\{X_i\}_{i\in I}$, denoted by $\prod_{i\in I} X_i$.

(5.3) As a warm up, let us write the universal property of direct sums. By definition $h^{\mathcal{X}} \xrightarrow[\psi]{\sim} \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i\in I} X_i, -)$

(if exists!)

i.e. there are natural bijections

$$h^{\mathcal{X}}(Y) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y) \xrightarrow{\psi_Y} \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{i\in I} X_i, -)$$

(1) Set $Y = \bigoplus_{i\in I} X_i$. The components of $\psi^{-1}_{\bigoplus_{i\in I} X_i}(\mathrm{Id}_{\bigoplus_{i\in I} X_i})$

give us morphisms

$$X_i \xrightarrow{\varphi_i} \bigoplus_{j\in I} X_j$$

(2) $\forall\, Y$ and morphisms $g_i : X_i \to Y$,

$\exists!\; g : \bigoplus_{j \in I} X_j \to Y$ (namely $\psi_Y((g_i)_{i\in I})$) s.t.

$g \circ \varphi_i = g_i$ (naturality of the natural iso. ψ)

Traditionally, direct sum is defined by (1) and (2).

e.g. $\mathcal{C} = \mathcal{Ab}$, category of abelian groups. $\{G_i\}_{i \in I}$ a set of objects of $\mathcal{C}$. Set $G = \{ (g_i \in G_i)_{i\in I} \mid g_j = 0$ for all but finitely many j's $\}$

Then $G = \bigoplus_{i \in I} G_i$ (i.e. G represents the functor $h^{\{G_i\}_{i\in I}}$, and hence, up to a unique iso., is <u>the</u> direct sum as defined in (5.2) above).

Similarly $\tilde{G}$ = cartesian product of G_i's ($= \prod_{i\in I} G_i$ as a set, with componentwise group operation) represents the contravariant functor $h_{\{G_i\}_{i\in I}}$.

(5.4) Direct sums / products need not exist. There existence of direct sums / products is a property of the category in question. For example, infinite direct sums (and products) do not exist in the category of finite-dim'l K-vector spaces.

Direct Sum in the category of sets = disjoint union (homework)

Remarks. — (1) $F(\bigoplus_{i\in I} X_i)$ need not equal $\bigoplus_{i\in I} F(X_i)$. Example

$$\mathcal{Ab} \xrightarrow[\text{forgetful functor}]{F} \underline{\text{Sets}}$$

. Direct sum of $\{G_i\}_{i\in I}$ as abelian groups, is not same as their direct sum as sets. (or even their direct sum in the category of all groups).

(2) Most examples of categories which do not have (infinite) direct sums arise when the category is not "big enough". (eg. Vect_K^{fd} vs. Vect_K - infinite direct sums exist in the later but not in the former).

(Exercise. - Direct sums & products do not exist in the category of fields.)

(5.5) $\bigoplus_{i\in I}$ and $\prod_{i\in I}$ as covariant functors. –

Let $\mathcal{C}$ be a category and I be a set. Assume $\bigoplus_{i\in I} X_i$ exists for any subset $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$.

Consider two such sets $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$; and assume we are given $f_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y_i)$ $\forall i \in I$.

Using canonical morphisms $\varphi_j : Y_j \to \bigoplus_{i\in I} Y_i$ we get $\varphi_j \circ f_j : X_j \to \bigoplus_{i\in I} Y_i$ $\forall j \in I$. Thus, by defn of $\bigoplus_{i\in I} X_i$

$\exists !$ morphism $\bigoplus_{i\in I} X_i \xrightarrow{f = \bigoplus_{i\in I} f_i} \bigoplus_{i\in I} Y_i$ s.t. $\forall j \in I$

$$\begin{tikzcd}
X_j \arrow[r, "f_j"] \arrow[d, "\psi_j"'] & Y_j \arrow[d, "\varphi_j"] \\
\bigoplus_{i\in I} X_i \arrow[r, "f"'] & \bigoplus_{i\in I} Y_i
\end{tikzcd}$$

commutes.

This can be rephrased as saying that we have a covariant functor $\bigoplus_{i\in I} : \mathcal{C}^I \longrightarrow \mathcal{C}$

where $\mathcal{C}^I$ is the product category: objects $\{X_i\}_{i\in I}$

$$\mathrm{Hom}_{\mathcal{C}^I}\left(\{X_i\}_{i\in I}, \{Y_i\}_{i\in I}\right) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_i)$$

Prop. – If $f_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y_i)$ is surjective $\forall i \in I$, then so is $\bigoplus_{j\in I} f_j : \bigoplus_{j\in I} X_j \longrightarrow \bigoplus_{j\in I} Y_j$.

Proof – $f = \bigoplus f_j$ is surjective, iff $\forall Z \in \mathcal{C}$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(\bigoplus Y_j, Z) & \xrightarrow{-\circ f} & \mathrm{Hom}_{\mathcal{C}}(\bigoplus X_j, Z) \\ \| & & \| \\ \prod_{j\in I} \mathrm{Hom}_{\mathcal{C}}(Y_j, Z) & \xrightarrow[(-\circ f_j)]{} & \prod_{j\in I} \mathrm{Hom}_{\mathcal{C}}(X_j, Z) \end{array}$$

is one-one

Thus if $\forall j \in I$, $\mathrm{Hom}_{\mathcal{C}}(Y_j, Z) \xrightarrow[-\circ f_j]{} \mathrm{Hom}_{\mathcal{C}}(X_j, Z)$ is one-one (i.e. f_j is surjective) then so is $-\circ f$. $\square$

(5.6) f_i injective $\forall i \in I$ does <u>not</u> imply that $\bigoplus_{i\in I} f_i$ is injective; Analogously, f_i injective $\forall i \in I$ $\Rightarrow \prod_{i\in I} f_i$ injective [such counterexamples are beyond the scope] (but not for surjective).