Lecture 6

(6.0) Recall that last time we defined direct sums and direct products in an arbitrary category $\mathcal{C}$. In almost similar fashion one can define direct and inverse limits.

(6.1) Let us fix a preordered set I. That is, there is a relation $\leq$ on $I \times I$ (recall: a relation is a subset of $I \times I$: we use the notation $i \leq j$ to mean (i,j) is in this subset).

$\leq$ satisfies: $\begin{cases} i \leq j \leq k \Rightarrow i \leq k & \text{(transitivity)} \\ i \leq i \quad \forall\, i \in I \end{cases}$

Note. Partial ordering has an additional axiom: $i \leq j \ \&\ j \leq i \Rightarrow i = j$ which is not imposed on preorder. This is not very important for us, and for all practical purposes, I will assume $(I, \leq)$ is a partially ordered set.

An <u>inverse system</u> on $(I, \leq)$ with values in $\mathcal{C}$ consists of the following data:

$$X_i \in \mathcal{C} \quad \forall\, i \in I$$

$$\varphi_{ij} : X_j \to X_i \quad \forall\, i \leq j$$

Subject to two conditions:

$$\varphi_{ii} = \mathrm{Id}_{X_i} \quad \forall\, i \in I$$

$$\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk} \quad \forall\, i \leq j \leq k$$

We can consider a (contra.) functor, for a given inverse system $\mathcal{X} = (\{X_i\}, \{\varphi_{ij}\})$, as follows:

$$h_{\mathcal{X}} : \mathcal{C} \longrightarrow \underline{\text{Sets}}$$

$$Y \longmapsto h_{\mathcal{X}}(Y) = \{(g_i : Y \to X_i)_{i \in I} \text{ such that } \forall i \leq j \quad \varphi_{ij}\, g_i = g_i\}$$

$$\left[\text{Note } h_{\mathcal{X}}(Y) \subset \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i)\right]$$

Definition.- If $h_{\mathcal{X}}$ is representable, an object representing it is called <u>the inverse limit</u> of the inverse system $\mathcal{X} = (\{X_i\}, \{\varphi_{ij}\})$, sometimes denoted by $\varprojlim_I X_i$.

(6.2) Let us spell out the universal property hidden in the definition above. Assume $X = \varprojlim_I X_i$ exists.

That is, we have natural isomorphism of functors

$$\psi : h_{\mathcal{X}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, X)$$

(i) $\psi_X : h_{\mathcal{X}}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, X)$

$$\{f_i : X \to X_i\}_{i \in I} \longleftrightarrow \mathrm{Id}_X$$

Thus X comes equipped with morphisms

$$f_i : X \to X_i \quad (\forall i \in I) \quad \text{s.t.} \quad \varphi_{ij} f_j = f_i \quad \forall i \leq j.$$

(ii) $$\begin{array}{ccc} \psi_Y : h_{\mathcal{X}}(Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(Y, X) \\ \psi & & \psi \\ (g_i : Y \to X_i) & \longmapsto & g \end{array}$$. That is,

$\forall\, Y \in \mathcal{C}$ and morphism $g_i : Y \to X_i$ s.t. $\varphi_{ij}\, g_j = g_i$ $(\forall\, i \leq j)$

$\exists!\; g : Y \to X$ s.t. $g_i = f_i \circ g$

(6.3) In concrete examples, $\varprojlim$ of an inverse system is constructed as a subset of the direct product.

e.g. $\mathcal{C} = \underline{\mathrm{Gps}}$ (category of groups).

$(I, \leq)$ an arbitrary partially ordered set

(G_i, φ_{ij}) an inverse system of groups (on $(I, \leq)$).

Let $G \subset \prod_{i \in I} G_i$ consist of elements $(\sigma_i)_{i \in I}$

s.t. $\forall\, i \leq j$, $\varphi_{ij}(\sigma_j) = \sigma_i$.

Prop. — $G \simeq \varprojlim_{I} G_i$ (has group structure, so is in the category $\underline{\mathrm{Gps}}$).

Proof. — $\prod_{i \in I} G_i$ has group structure = componentwise

Claim. - $G \subset \prod_{i \in I} G_i$ is a subgroup.

Pf. of the claim. — $e = (e_i)_{i \in I} \in G$ (here $e_i \in G_i$ is the unit)

because $\varphi_{ij} : G_j \to G_i$ is a gp. hom.

Let (σ_i), $(\tau_i) \in G$. Then $\varphi_{ij}(\sigma_j^{-1}) = \sigma_i^{-1}$

$$\Rightarrow (\sigma_i^{-1})_{i \in I} \in G$$

$$\varphi_{ij}(\sigma_j \cdot \tau_j) = \varphi_{ij}(\sigma_j)\,\varphi_{ij}(\tau_j) = \sigma_i \tau_i \Rightarrow (\sigma_i \tau_i)_{i \in I} \in G.$$

(end of the pf. of the claim).

(i) Natural homomorphisms

$$f_i : \begin{array}{ccc} G & \longrightarrow & G_i \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ (\sigma_j)_{j \in I} & \longmapsto & \sigma_i \end{array}$$

(ii) Given gp. homs. $g_i : H \to G_i$ s.t. $\varphi_{ij}\, g_j = g_i$ $\forall i \leq j$

we get $\begin{array}{ccc} H & \longrightarrow & \prod_{i \in I} G_i \\ \cup\!\!\!\shortmid & & \\ \sigma' & \longmapsto & (g_i(\sigma'))_{i \in I} \end{array}$. We claim the image of this map lands in G. That is,

$\forall\, i \leq j \qquad \varphi_{ij}(g_j(\sigma')) = g_i(\sigma')$ which is true by our hypothesis on $\{g_i : H \to G_i\}_{i \in I}$.

This is the unique map so that
$$\begin{tikzcd} H \arrow[r] \arrow[dr, "g_i"'] & G \arrow[d, "f_i"] \\ & G_i \end{tikzcd}$$
commutes $\forall\, i \in I$. $\square$

The same construction goes through in the categories of R-modules, Rings, and many others.

(6.4) An example. $\mathcal{C}$ = CommRings (commutative ring with unit, morphisms are ring hom mapping unit to unit).

$\forall\, n \geq 1$, let $R_n = K[x]/(x^n)$ (K : a field)

$\forall\, n \leq m$, $\varphi_{nm} : R_m \longrightarrow R_n$ natural projection
$$K[x]/(x^m) \to K[x]/(x^n) \qquad (x \mapsto x).$$

Compute $\varprojlim R_n$. Answer: $\displaystyle\varprojlim_n \frac{K[X]}{(X^n)} = K[\![X]\!]$

Ring of formal power series $\displaystyle\sum_{\ell=0}^{\infty} a_\ell x^\ell$ $(a_0, a_1, \dots \in K)$

Proof. (1) Morphisms $f_n : K[\![x]\!] \to K[x]/(x^n)$

$$\sum_{l=0}^{\infty} a_l x^l \longmapsto \sum_{l=0}^{n-1} a_l x^l$$

(2) Given ring homomorphisms $g_n : R \to K[x]/(x^n)$

such that
$$\begin{tikzcd} R \arrow[r, "g_m"] \arrow[dr, "g_n"'] & K[x]/(x^m) \arrow[d, "\varphi_{nm}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & K[x]/(x^n) \end{tikzcd} \qquad \forall\, n \le m.$$

define $g : R \to K[\![x]\!]$ by $g(r) = \sum_{l=0}^{\infty} a_l(r)\, x^l$

where $a_l(r)$ = coeff of x^l in $g_m(r)$ for any $m > l$.

(does not depend on m, because of the commutativity of the diagram above).

The reader should verify that this is the only ring hom. such that $g_n = f_n \circ g$ $\forall\, n \ge 1$. $\square$

(6.5) Direct limit.

Again consider a preordered set $(I, \leq)$. A <u>direct system</u> over I, with values in $\mathcal{C}$, is the data of:

$$X_i \in \mathcal{C} \quad \forall i \in I \qquad \text{and} \quad \psi_{ji} : X_i \longrightarrow X_j \quad \forall i \leq j$$

subject to: $\psi_{ii} = \mathrm{Id}_{X_i}$; $\psi_{kj} \circ \psi_{ji} = \psi_{ki} \quad \forall i \leq j \leq k$

We often add the hypothesis that $(I, \leq)$ is "right directed" i.e. $\forall\, i, j \in I$, $\exists\, k \in I$ s.t. $i \leq k$ and $j \leq k$.

We will see exactly why this hypothesis is needed later.

Define the (covariant) functor, for a given direct system $\mathcal{X} = (\{X_i\}, \{\psi_{ji}\})$ as:

$$h^{\mathcal{X}} : \mathcal{C} \longrightarrow \underline{\text{Sets}}$$

$\forall\, Y \in \mathcal{C}$, $\quad h^{\mathcal{X}}(Y) \subset \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$ consists of

$$\{f_i : X_i \to Y\}_{i \in I} \quad \text{s.t. } \forall\, i \leq j, \quad f_j \circ \psi_{ji} = f_i .$$

Definition. – If h^* is representable, the object representing it is called the <u>direct limit</u> of the direct system $(\{X_i\}, \{\varphi_{ji}\})$, denoted by $\varinjlim_{I} X_i$.

(6.6) Unfolding the definition. $X = \varinjlim X_i$, if exists, satisfies the following universal property. (compare w/ §6.2)

(1) There are morphisms $\{f_i : X_i \to X\}_{i \in I}$ s.t.

$$\forall\, i \leq j, \quad f_j \varphi_{ji} = f_i.$$

(2) $\forall\, Y \in \mathcal{C}$ and morphisms $g_i : X_i \to Y$ s.t

$\forall\, i \leq j$, $g_j \varphi_{ji} = g_i$, $\exists!\; g : X \to Y$ such that

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] \arrow[dr, "g_i"] & \\ X \arrow[r, "g"'] & Y \end{tikzcd} \qquad (g_i = g \circ f_i \;\; \forall\, i \in I)$$