Lecture 7

(7.0) Recall: last time we defined inverse and direct limits. Schematically the construction went as follows.

Input: a preordered set $(I, \leq)$, a category $\mathcal{C}$.

- directed system: $\left(\{X_i\}_{i \in I}, \{\psi_{ji}\}_{i \leq j} \right)$

  $[X_i \in \mathcal{C} \ (\forall i) \ ; \ \psi_{ii} = \mathrm{Id}_{X_i} \ ; \ \psi_{kj}\psi_{ji} = \psi_{ki} \ \forall i \leq j \leq k]$

- functor: $h^{\mathcal{X}} : \mathcal{C} \longrightarrow \underline{\text{Sets}}$

  $[h^{\mathcal{X}}(Y) \subset \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y) \ ; \ (g_i : X_i \to Y)_{i \in I} \in h^{\mathcal{X}}(Y)$ if

  $g_j \psi_{ji} = g_i \quad \forall i \leq j]$

Output.- a question. Is $h^{\mathcal{X}}$ representable?

If the answer is yes, the object representing it is $\varinjlim_I X_i$. (defined to be)

If the answer is no, there is no direct limit of the directed system $(\{X_i\} ; \{\psi_{ji}\})$.

(7.1) We will always assume $(I, \leq)$ is right directed, when talking about direct limits.

Example from complex analysis - (germs of holomorphic fns near a point). For $z \in \mathbb{C}$, a (local) ring is defined in complex function theory, denoted by $\mathcal{O}_z$, as:

$$\mathcal{O}_z = \frac{\left\{ (f, \mathcal{U}) \text{ where } z \in \mathcal{U} \underset{\text{open}}{\subseteq} \mathbb{C}, \text{ and } \underset{\text{is holomorphic}}{f : \mathcal{U} \to \mathbb{C}} \right\}}{\text{equiv. rel}^n \quad (f, \mathcal{U}) \sim (f', \mathcal{U}') \text{ if } \exists\, V \underset{\text{open}}{\subseteq} \mathcal{U} \cap \mathcal{U}' \\ z \in V, \text{ and } f|_V = f'|_V}$$

- Directed System. Take $I = \{\text{open subsets of } \mathbb{C} \text{ containing } z\}$

preorder = reverse inclusion $(V \subset \mathcal{U} \leftrightarrow \mathcal{U} \leq V)$

Right directed because finite intersection of open sets is open.

To each $\mathcal{U} \in I$, $\mathcal{O}(\mathcal{U}) = \{\text{all hol. fns. } \mathcal{U} \to \mathbb{C}\}$
(commutative ring)

$\forall\ \mathcal{U} \leq V$ (i.e. $V \subset \mathcal{U}$), $\quad \mathcal{O}(\mathcal{U}) \longrightarrow \mathcal{O}(V)$ restriction map

$f \longmapsto f|_V$

$$\mathcal{O}_z = \varinjlim \mathcal{O}(\mathcal{U})$$

(7.2) Typically, direct limits are constructed as quotients of direct sums, by an equivalence relation. Let us see this in the category $\underline{Ab}$.

Prop. – Direct limits exist in the category $\underline{Ab}$.

Proof. – Let $(I, \leq)$ be a right directed preordered set and let $\left(\{G_i\}_{i \in I}, \{\psi_{ji}\}_{i \leq j}\right)$ be a directed system of abelian groups. Define

subgroup gen. by $\sigma_i - \psi_{ji}(\sigma_i)$

$$G := \bigoplus_{i \in I} G_i \Bigg/ \left\langle \begin{matrix} \psi_{ji}(\sigma_i) - \sigma_i \;\; \forall\, i \leq j \\ \sigma_i \in G_i, \cancel{\sigma_j \in G_j} \end{matrix} \right\rangle$$

Morphisms. $G_i \xrightarrow{f_i} G$ are given by

$$\begin{tikzcd} G_i \arrow[r, "\text{inclusion}"] \arrow[rr, bend right, "f_i"'] & \bigoplus_{j \in I} G_j \arrow[r, "\text{quotient}"] & G \end{tikzcd}$$

To show. – (i) $f_j \psi_{ji} = f_i \quad \forall\, i \leq j$

Pf. – $\forall\, \sigma_i \in G_i$, we have two elements of $\bigoplus_{j \in I} G_j$, namely

$$\begin{array}{c} \sigma_i \in G_i \hookrightarrow \bigoplus_{k\in I} G_k \\ \sigma_j = \psi_{ji}(\sigma_i) \in G_j \hookrightarrow \bigoplus_{k\in I} G_k \end{array}$$

Their images in $\bigoplus_{k \in I} G_k \twoheadrightarrow G$ are the same by definition. Thus $f_j(\psi_{ji}(\sigma_i)) = f_i(\sigma_i) \quad \forall\, i \leq j$, $\sigma_i \in G_i$.

(ii) $\forall\, H \in \underline{Ab}$ and $g_i : G_i \to H$ gp. hom. s.t. $g_j \psi_{ji} = g_i$; define $\bigoplus_{k \in I} G_k \xrightarrow{\tilde{g} = \sum_{k\in I} g_k} H$ (by universal property of $\bigoplus_{k\in I} G_k$). Claim. $\forall\, \sigma \in G_i$,

$\tilde{g}(\sigma - \psi_{ji}(\sigma)) = 0$ (because it is equal to $g_i(\sigma) - g_j(\psi_{ji}(\sigma)) = 0$). Thus we get $g : G \to H$ unique gp. hom. s.t.

$$\begin{tikzcd} G_i \arrow[d, "f_i"'] \arrow[dr, "g_i"] & \\ G \arrow[r, "g"'] & H \end{tikzcd}$$

commutes $\forall\, i \in I$ □

(7.3) Inverse limit vs. direct product: if $(I, \leq)$ is totally disconnected (i.e. $i \leq j \iff i = j$) then

$$\varprojlim_{I} = \prod_{i \in I}.$$

This is clear from the definitions.

However for ~~inverse~~ direct limits, a little care is needed.

Let J be a set. Define I = set of finite subsets of J, ordered by inclusion. This is a right directed set (union of two finite sets is a finite set).

$$\{X_j\}_{j \in J} \longrightarrow \{X_\alpha\}_{\alpha \in I}$$

finite subset of J

a set of objects of a category $\mathcal{C}$

$$X_\alpha := \bigoplus_{j \in \alpha} X_j$$

$$\alpha \leq \beta \rightsquigarrow X_\alpha \xrightarrow{\psi_{\beta\alpha}} X_\beta$$

(i.e., $\alpha \subseteq \beta$)

$\psi_{\beta\alpha}$ is defined by using morphisms $f_j : X_j \to \bigoplus_{k \in \beta} X_k$ $\forall\, j \in \alpha$

which by definition of direct sum, extend to $\bigoplus_{j \in \alpha} X_j \to \bigoplus_{k \in \beta} X_k$

In most concrete categories, (homework)

$$\bigoplus_{j \in J} X_j \quad \text{end up being isomorphic to} \quad \varinjlim_{I} X_\alpha .$$

~~(7.4)~~