

Lecture 7

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(7.0) Recall: last time we defined inverse and direct limits. Schematically the construction went as follows.

Input: a preordered set (I, \leq) , a category \mathcal{C} .

• directed system: $(\{X_i\}_{i \in I}, \{\psi_{ji}\}_{i \leq j})$

$$[X_i \in \mathcal{C} (\forall i) ; \psi_{ii} = \text{Id}_{X_i} ; \psi_{kj} \psi_{ji} = \psi_{ki} \quad \forall i \leq j \leq k]$$

• functor: $h^{\mathcal{X}} : \mathcal{C} \rightarrow \underline{\text{Sets}}$

$$[h^{\mathcal{X}}(Y) = \prod_{i \in I} \text{Hom}_{\mathcal{C}}(X_i, Y) ; (g_i : X_i \rightarrow Y)_{i \in I} \in h^{\mathcal{X}}(Y) \text{ if}$$

$$g_j \psi_{ji} = g_i \quad \forall i \leq j]$$

Output. - a question. Is $h^{\mathcal{X}}$ representable?

defined to be

If the answer is yes, the object representing it is $\varinjlim_I X_i$.

If the answer is no, there is no direct limit of the

directed system $(\{X_i\}; \{\psi_{ji}\})$.

(7.1) We will always assume (I, \leq) is right directed, when talking about direct limits.

(2)

Example from complex analysis - (germs of holomorphic fns near a point). For $z \in \mathbb{C}$, a (local) ring is defined in complex function theory, denoted by \mathcal{O}_z , as:

$$\mathcal{O}_z = \left\{ (f, U) \text{ where } z \in U \underset{\text{open}}{\subset} \mathbb{C}, \text{ and } f: U \rightarrow \mathbb{C} \text{ is holomorphic} \right\}$$

$$\text{equiv. rel.}^n \quad (f, U) \sim (f', U') \text{ if } \exists V \underset{\text{open}}{\subset} U \cap U' \\ z \in V, \text{ and } f|_V = f'|_V$$

• Directed System. Take $I = \{ \text{open subsets of } \mathbb{C} \text{ containing } z \}$
preorder = reverse inclusion $(V \subset U \leftrightarrow U \leq V)$

Right directed because finite intersection of open sets is open.

To each $U \in I$, $\mathcal{O}(U) = \{ \text{all hol. fns. } U \rightarrow \mathbb{C} \}$
(commutative ring)

$\forall U \leq V$, $\mathcal{O}(U) \rightarrow \mathcal{O}(V)$ restriction map
(i.e. $V \subset U$) $f \longmapsto f|_V$

$$\mathcal{O}_z = \varinjlim \mathcal{O}(U)$$

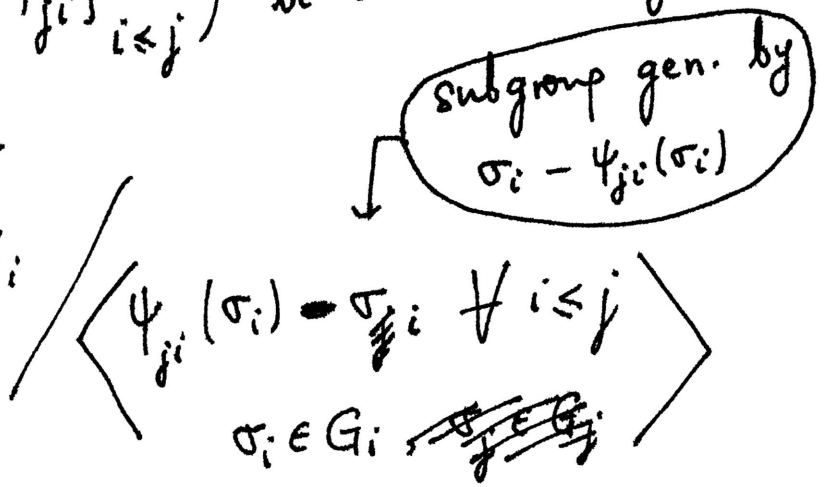
(7.2) Typically, direct limits are constructed as quotients of direct sums, by an equivalence relation. Let us see this in the category Ab.

Prop. - Direct limits exist in the category Ab.

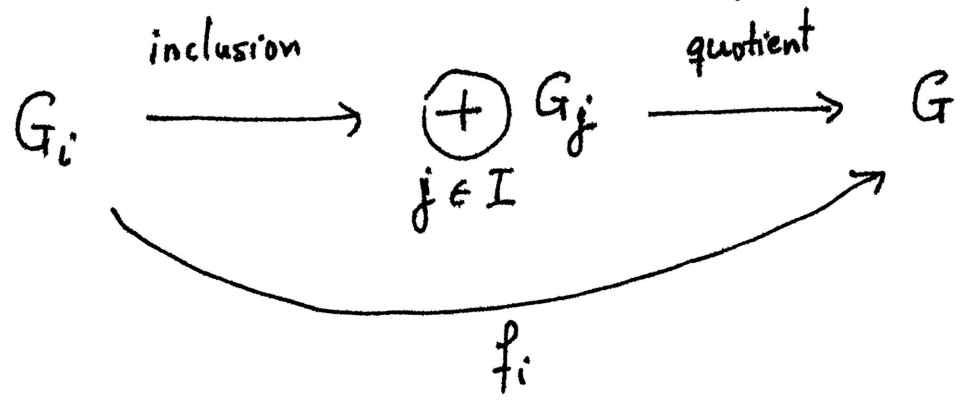
Proof. - Let (I, \leq) be a right directed preordered set and let $(\{G_i\}_{i \in I}, \{\psi_{ji}\}_{i \leq j})$ be a directed system

of abelian groups. Define

$$G := \bigoplus_{i \in I} G_i$$



Morphisms. $G_i \xrightarrow{f_i} G$ are given by



To show. - (i) $f_j \psi_{ji} = f_i \quad \forall i \leq j$

Pf. - $\forall \sigma_i \in G_i$, we have two elements $\frac{f}{\#}$ of

$$\bigoplus_{j \in I} G_j, \text{ namely } \sigma_i \in G_i \hookrightarrow \bigoplus_{k \in I} G_k$$

$$\sigma_j = \psi_{ji}(\sigma_i) \in G_j \quad \swarrow$$

Their images in $\bigoplus_{k \in I} G_k \longrightarrow G$ are the same by

definition. Thus $f_j(\psi_{ji}(\sigma_i)) = f_i(\sigma_i) \quad \forall i \leq j$
 $\sigma_i \in G_i$

(ii) $\forall H \in \underline{Ab}$ and $g_i : G_i \rightarrow H$ gp. hom. s.t.

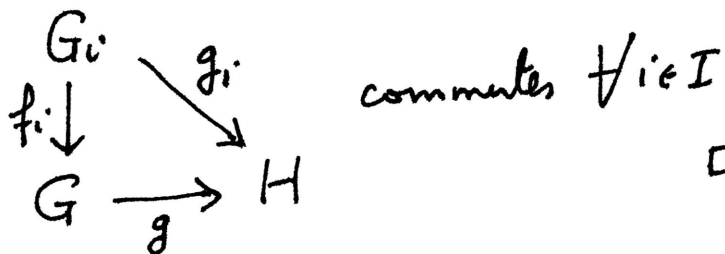
$$g_j \psi_{ji} = g_i ; \text{ define } \bigoplus_{k \in I} G_k \xrightarrow{\tilde{g} = \sum_{k \in I} g_k} H \text{ (by}$$

universal property of $\bigoplus_{k \in I} G_k$). Claim. $\forall \sigma \in G_i$,

$$\tilde{g}(\sigma - \psi_{ji}(\sigma)) = 0 \quad (\text{because it is equal to}$$

$$g_i(\sigma) - g_j(\psi_{ji}(\sigma)) = 0). \text{ Thus we get } g : G \rightarrow H$$

unique gp. hom. s.t.



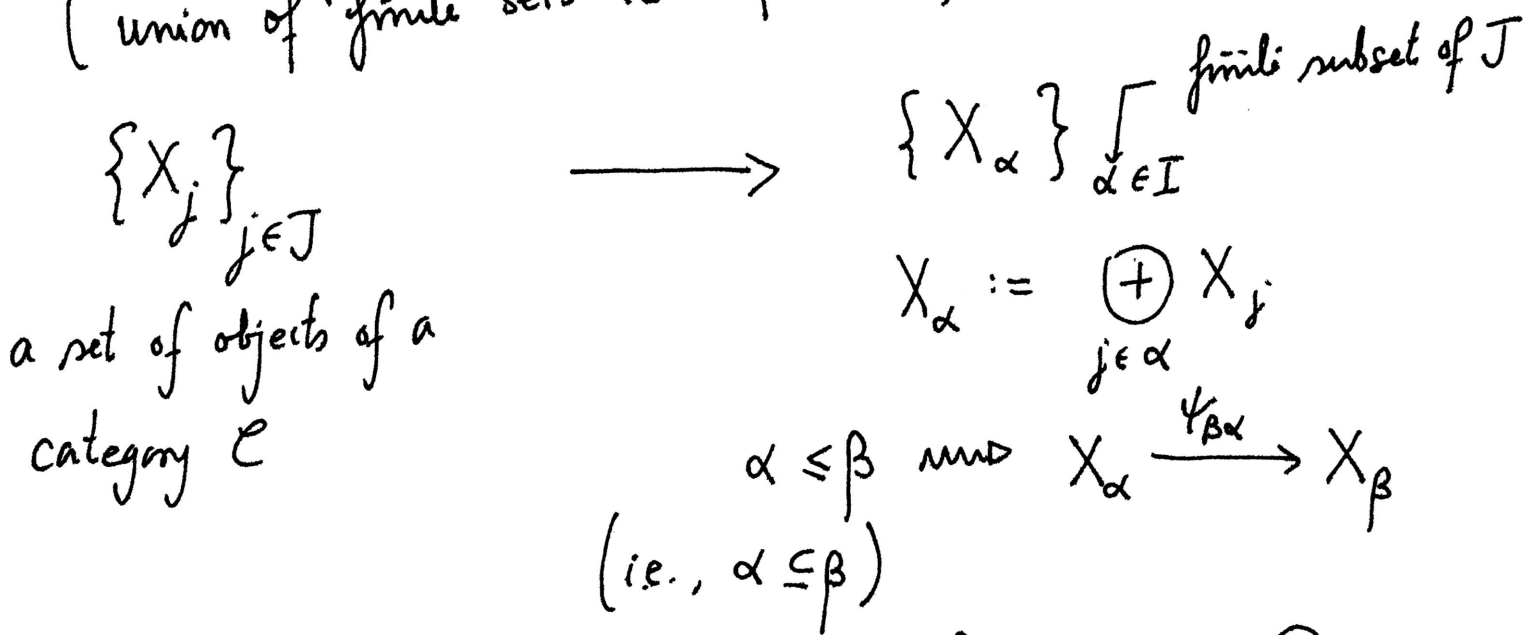
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(7.3) Inverse limit vs. direct product: if (I, \leq) is totally disconnected (i.e. $i \leq j \iff i = j$) then

$\lim_{\leftarrow I} = \prod_{i \in I}$. This is clear from the definitions.

However for ~~inverse~~ ^{direct} limits, a little care is needed.

Let J be a set. Define $I =$ set of finite subsets of J , ordered by inclusion. This is a right directed set (union of ^{two} finite sets is a finite set).



$\psi_{\beta\alpha}$ is defined by using morphisms $f_j : X_j \rightarrow \bigoplus_{k \in \beta} X_k$ $\forall j \in \alpha$

which by definition of direct sum, extend to $\bigoplus_{j \in \alpha} X_j \rightarrow \bigoplus_{k \in \beta} X_k$

In most concrete categories, (homework)

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$\bigoplus_{j \in J} X_j$ end up being isomorphic to $\varinjlim X_\alpha$.

~~(7.4)~~