

Lecture 8

①

(8.0) Definition. - Let \mathcal{C} be a category. We say \mathcal{C} is an

additive category if

(i) $\forall X, Y \in \mathcal{C}$, $\text{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group so that

the composition maps $\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$

$$(f, g) \longmapsto g \circ f$$

are (\mathbb{Z}) -bilinear. $[g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2; (g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f]$

(ii) There is a zero object, denoted by $0_{\mathcal{C}}$. (see Lecture 0 p.8)

Recall that, in the context of ^{an} additive category, this means

$\forall X \in \mathcal{C}$, $\text{Hom}_{\mathcal{C}}(0_{\mathcal{C}}, X)$ and $\text{Hom}_{\mathcal{C}}(X, 0_{\mathcal{C}})$ are both

trivial abelian groups.

(iii) Finite direct sums and products exist.

Examples. (a) Most categories out of the realm of algebra are not additive. e.g., categories of topological spaces, smooth manifolds. Within algebra categories of all groups, commutative rings are also not additive.

(b) Most typical examples of additive categories are of the

form $R\text{-mod} = \text{category of left } R\text{-modules}$
($\text{mod-}R = \text{category of right } R\text{-modules}$)

where R is a ring (with $1, 0 \in R$ distinct). Recall:

a left R -module is an abelian group M together with a left action of R on M , i.e. a (unital) ring homomorphism

$$R \xrightarrow{\lambda} \text{End}_{\text{gps}}(M)$$

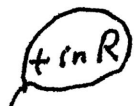
Shorthand for scalar multiplication: $\forall r \in R; m \in M$,
 $r \cdot m := \lambda(r)(m)$. Then λ is a unital ring hom.

translates to: (i) $0 \cdot m = 0 \quad \forall m \in M$



(ii) $1 \cdot m = m \quad \forall m \in M$

(iii) $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$
& $(r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m) \quad \forall r_1, r_2 \in R; m \in M$



(iv) $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2 \quad \forall r \in R; m \in M$.

$\text{Hom}_{R\text{-mod}}(M, N) = \left\{ f : M \rightarrow N \text{ gp. hom such that} \right.$
 $\left. \begin{matrix} f(r \cdot m) = r \cdot f(m) \\ \forall r \in R; m \in M \end{matrix} \right\}$

\cup

f_1, f_2 Define. $f_1 + f_2 : M \rightarrow N$ by $(f_1 + f_2)(m) = f_1(m) + f_2(m)$

Thus $\text{Hom}_{R\text{-mod}}(M, N)$ has a str. of an abelian

group ($0 \in \text{Hom}_{R\text{-mod}}(M, N)$ is the map $m \mapsto 0 \in N \quad \forall m \in M$)

Easy check. - composition is bilinear.

Zero object of $R\text{-mod}$. - $0_{R\text{-mod}}$ is trivial abelian group.

with the only possible R -action: $r \cdot 0 = 0 \quad \forall r \in R$.

Finite direct sums (= products). - $M, N \in R\text{-mod}$

$$M \oplus N = \{ (m, n) : m \in M, n \in N \} \text{ with componentwise}$$

$$(\text{= } M \times N) \quad \text{operations: } \begin{cases} (m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2) \\ r \cdot (m, n) = (r \cdot m, r \cdot n) \end{cases}$$

[Recall: for $\text{mod-}R$, a right R module is again an abelian group M and a ~~right~~ ring hom $R^{\text{op}} \xrightarrow{\rho} \text{End}_{\text{gp}}(M)$.

Scalar product in this case is written on the right:

$m \cdot r = \rho(r)(m)$. The only change is in (iii) of page 2:

$m \cdot (r_1 r_2) = (m \cdot r_1) \cdot r_2$ because $\rho(r_1 r_2) = \rho(r_2) \rho(r_1)$]

(8.1) Lemma. — In an additive category, finite direct sums and direct products are isomorphic.

Proof. Let \mathcal{C} be an additive category and I be a set.

Let $\{X_i\}_{i \in I}$ be a subset of the class of objects of \mathcal{C} . Assume

$X := \bigoplus_{i \in I} X_i$ and $\hat{X} = \prod_{i \in I} X_i$ exist in \mathcal{C} . (We will impose

finiteness of I , later when it is needed). Thus, we are given

morphisms:

$$X_i \xrightarrow{f_i} \bigoplus_{j \in I} X_j = X \quad \text{and} \quad \hat{X} = \prod_{j \in I} X_j \xrightarrow{\hat{f}_i} X_i$$

$\forall i, j$; let $\delta_{ji} : X_i \rightarrow X_j$ be the morphism $\begin{cases} 0 & \text{if } i \neq j \\ \text{Id}_{X_i} & \text{if } i = j \end{cases}$

Step 1. For a fixed $i \in I$, the collection of morphisms

$\{\delta_{ji} : X_i \rightarrow X_j\}_{j \in I}$ define, by the universal property of

direct product, a morphism $\hat{f}_i : X_i \rightarrow \hat{X}$.

Similarly the collection of morphisms $\{\delta_{ij} : X_j \rightarrow X_i\}_{j \in I}$

defines a morphism $\bigoplus_{j \in I} X_j = X \xrightarrow{f^i} X_i$

Because of the universal property we get

$$f^j f_i = \delta_{ji} \quad \text{and} \quad \hat{f}^j \hat{f}_i = \delta_{ji}.$$

[eg. f^j is the unique map st. $\bigoplus_{k \in I} X_k = X \xrightarrow{f^j} X_j$

s.t. $\forall i \in I$

$$\begin{array}{ccc} X_i & \xrightarrow{\delta_{ji}} & X_j \\ f_i \downarrow & & \nearrow \\ X & \xrightarrow{f^j} & X_j \end{array} \quad \text{commutes.}$$

Step 2. $\alpha: \bigoplus_{i \in I} X_i = X \longrightarrow \hat{X} = \prod_{i \in I} X_i$ can be defined

either by $\cdot \left\{ X \xrightarrow{f^i} X_i \right\}_{i \in I} \xrightarrow{\text{U.P. of } \prod} X \xrightarrow{\alpha} \hat{X}$

or by $\cdot \left\{ X_i \xrightarrow{\hat{f}_i} \hat{X} \right\}_{i \in I} \xrightarrow{\text{U.P. of } \oplus} X \xrightarrow{\alpha} \hat{X}$

Check. - In either case, $\alpha: X \rightarrow \hat{X}$ is the unique morphism s.t.

$$\begin{array}{ccc} X_i & \xrightarrow{\delta_{ji}} & X_j \\ f_i \downarrow & & \uparrow \hat{f}^j \\ X & \xrightarrow{\alpha} & \hat{X} \end{array} \quad \text{commutes } \forall i, j \in I.$$

ie. $\hat{f}^j \alpha f_i = \delta_{ji} \quad \forall i, j \in I.$

Step 3. Now assume I is finite and define (6)

$$\beta: \hat{X} \rightarrow X \quad \text{as} \quad \beta = \sum_{i \in I} f_i \hat{f}^i$$

$\begin{array}{ccc} & \nearrow f_i & \\ \hat{f}_i \downarrow & & \\ X_i & & \forall i \in I \end{array}$

\uparrow
 sum in the abelian
 group $\text{Hom}_e(\hat{X}, X)$.

$X \xrightleftharpoons[\beta]{\alpha} \hat{X}$. We will now show that $\beta \alpha: X \rightarrow X$ is identity. I will leave the proof of $\alpha \beta = \text{Id}_{\hat{X}}$ which is exactly on the same lines.

$\beta \alpha: X \rightarrow X$. For every $i \in I$, we have

$$(\beta \alpha) \circ f_i = \sum_{j \in I} f_j \boxed{f^j \alpha f_i} = f_i$$

$\uparrow = \delta_{ji}$
(see prev. page)

set $Y = X$ & $(X_i \rightarrow Y) = f_i$

In the universal property of $\bigoplus_{j \in I} X_j$, there must be a unique

morphism $X \xrightarrow{g} Y$ s.t. $g \circ f_i = f_i \quad \forall i \in I$.

(x)

$g = \text{Id}_X$ and $g = \beta \alpha$ both satisfy this condition $\Rightarrow \text{Id}_X = \beta \alpha$ □

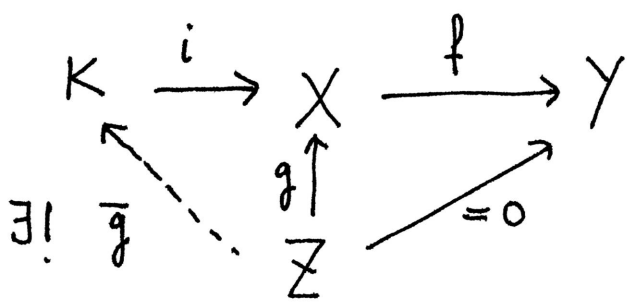
(8.2) Kernels and cokernels, ~~exist~~ - Again let \mathcal{C} be an additive category and let $f: X \rightarrow Y$ be a morphism.

Kernel of f is a pair $(K, i: K \rightarrow X)$ such that

- i is injective
- $f \circ i = 0$
- for every $Z \in \mathcal{C}$

and $g: Z \rightarrow X$ s.t. $f \circ g = 0$, there is a (unique)

$\bar{g}: Z \rightarrow K$ s.t. $g = i \circ \bar{g}$. Pictorially:

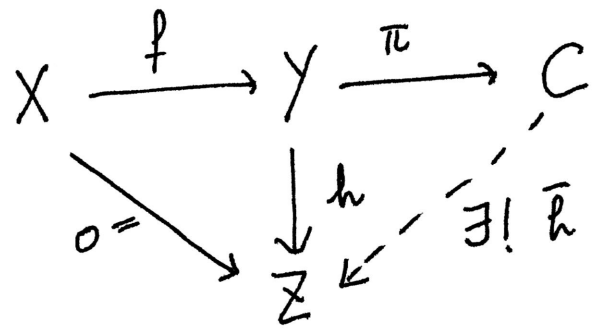


Cokernel of f is a pair $(C, \pi: Y \rightarrow C)$ s.t.

- π is surjective
- $\pi \circ f = 0$
- for every

$Z \in \mathcal{C}$ and morphism $h: Y \rightarrow Z$ s.t. $h \circ f = 0$, there

is a (unique) $\bar{h}: C \rightarrow Z$ s.t. $h = \bar{h} \circ \pi$.



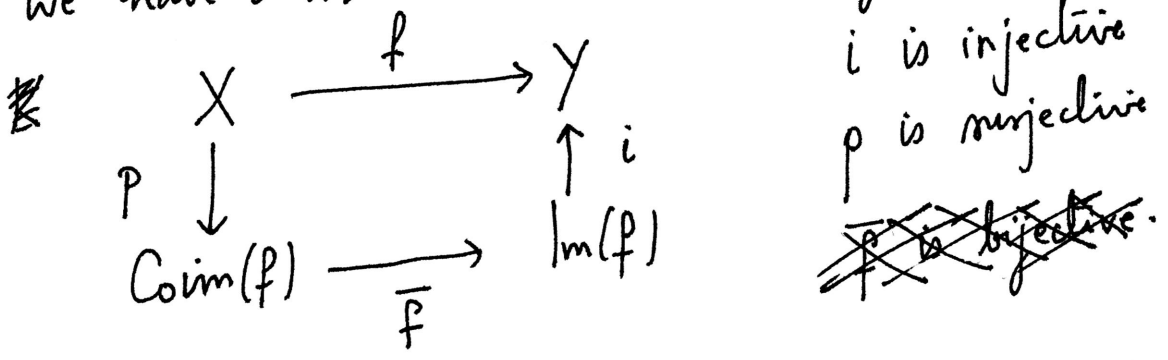
(8.2) Proposition. — Let \mathcal{C} be an additive category and let

$f: X \rightarrow Y$ be a morphism in \mathcal{C} . Then

- f is injective iff $\text{Ker}(f) = (0_{\mathcal{C}}, 0_{\mathcal{C}} \xrightarrow{\text{unique}} X)$
- f is surjective iff $\text{Coker}(f) = (0_{\mathcal{C}}, Y \rightarrow 0_{\mathcal{C}})$

Assume kernels and cokernels exist in \mathcal{C} . Define
 Image of $f = \text{Im}(f) := \text{Ker}(Y \xrightarrow[\text{Coker}(f)]{\pi} \mathcal{C})$ (similarly $\text{Coim}(f)$)

Then we have a ~~natural~~ commutative diagram



Proof. — (i) f is injective $\iff \text{Ker}(f) = 0$.

(\implies) $0_{\mathcal{C}} \xrightarrow{0} X \xrightarrow{f} Y$ composition is 0 ✓

Let $Z \xrightarrow{g} X$ be a morphism s.t. $f \circ g = 0$. Then the same is true for $Z \xrightarrow{0} X$. As f is injective, we get $g = 0$.

i.e. g factors through $0_{\mathcal{C}} \rightarrow X$.

(\Leftarrow) $\text{Ker}(f)$ (exists) and is $0_e \xrightarrow{0} X$. To prove:

f is injective. Let $Z \begin{matrix} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{matrix} X$ be two morphisms s.t.

$f \circ g_1 = f \circ g_2$, i.e. $f \circ (g_1 - g_2) = 0$. By defn of kernel

$g = g_1 - g_2$ must factor through $Z \begin{matrix} \xrightarrow{g} \\ \searrow \\ 0_e \end{matrix} X$; i.e. $g = 0$

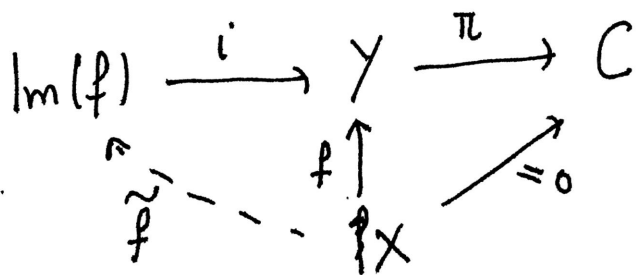
$\Rightarrow g_1 = g_2$.

(2) For the surjectivity case, use the same argument as above.

(3) Now let $X \xrightarrow{f} Y$ be any morphism in \mathcal{C} .

Let C be cokernel of f . That is, $X \xrightarrow{f} Y \xrightarrow{\pi} C$, $\pi \circ f = 0$.

By defn. of kernel $\exists X \rightarrow \text{Ker}(\pi) = \text{Im}(f)$ s.t.



i is injective

$f = \tilde{f} \circ i$

Now let $\text{Ker}(f) \xrightarrow{k} X$ be the kernel of f . Thus $f \circ k = 0$

$\Rightarrow i \circ \tilde{f} \circ k = 0 = i \circ (0) \Rightarrow \tilde{f} \circ k = 0$ by injectivity of i

This argument shows that $(\text{Ker}(f), k)$ is also kernel of \tilde{f}
 (the reader should fill in the details).

$$\text{Ker}(f) \xrightarrow{k} X \xrightarrow{\tilde{f}} \text{Im}(f) \quad \text{By defn. of } \textcircled{10}$$

$\underbrace{\hspace{10em}}_{=0}$

Coinage (= cokernel of $\text{Ker}(f) \xrightarrow{k} X$) we get

$$\text{Ker}(f) \xrightarrow{k} X \xrightarrow{p} \text{Cok}(f)$$

s.t. $\bar{f} \circ p = \tilde{f}$

and p is surjective.

$$\begin{array}{ccc} & \tilde{f} & \\ \searrow & \downarrow & \swarrow \\ 0 & \text{Im}(f) & \end{array}$$

□