Lecture 8

(8.0) Definition. – Let $\mathcal{C}$ be a category. We say $\mathcal{C}$ is an <u>additive category</u> if

(i) $\forall\, X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an abelian group so that the composition maps $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$

$$(f, g) \longmapsto g\circ f$$

are $(\mathbb{Z}\text{-})$ bilinear. $\left[g\circ(f_1+f_2) = g\circ f_1 + g\circ f_2 \,;\ (g_1+g_2)\circ f = g_1\circ f + g_2\circ f \right]$

$n(g\circ f) = (ng)\circ f = g\circ(nf)$, $\forall\, n \in \mathbb{Z}$

(ii) There is a zero object, denoted by $0_{\mathcal{C}}$. (see Lecture 0 p.8)

Recall that, in the context of an additive category, this means $\forall\, X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(0_{\mathcal{C}}, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, 0_{\mathcal{C}})$ are both trivial abelian groups.

(iii) Finite direct sums and products exist.

Examples. (a) Most categories out of the realm of algebra are <u>not</u> additive. e.g., categories of topoligical spaces, smooth manifolds. Within algebra categories of all groups, commutative rings <u>are also not additive</u>.

(b) Most typical examples of additive categories are of the form R-mod = category of left R-modules

(mod-R = category of right R-modules)

where R is a ring (with $1, 0 \in R$ distinct). Recall:

a left R-module is an abelian group M together with a left action of R on M, i.e. a (unital) ring homomorphism

$$R \xrightarrow{\lambda} \mathrm{End}_{gps}(M)$$

Shorthand for scalar multiplication: $\forall r \in R$; $m \in M$,

$r.m := \lambda(r)(m)$. Then λ is a unital ring hom.

translates to: (i) $0 \cdot m = 0 \quad \forall m \in M$

of R of M $+$ in R $+$ in M

(ii) $1 \cdot m = m \;\forall m \in M$ (iii) $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$

& $(r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m) \;\forall r_1, r_2 \in R;\ m \in M$

(iv) $r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2 \quad \forall r \in R;\ m \in M.$

$$\mathrm{Hom}_{R\text{-mod}}(M, N) = \left\{ f : M \to N \text{ gp. hom such that } f(r \cdot m) = r \cdot f(m) \quad \forall \begin{array}{l} r \in R; \\ m \in M \end{array} \right\}$$

$\Downarrow$

f_1, f_2 Define. $f_1 + f_2 : M \to N$ by $(f_1 + f_2)(m) := f_1(m) + f_2(m) \quad \forall m \in M$

Thus $\mathrm{Hom}_{R\text{-mod}}(M,N)$ has a str. of an abelian group. ($0 \in \mathrm{Hom}_{R\text{-mod}}(M,N)$ is the map $m \longmapsto 0 \in N \quad \forall m \in M$).

Easy check. – composition is bilinear.

Zero object of R-mod. – $\underset{R\text{-mod}}{0} \overset{\{0\}}{\underset{\|}{}}$ is trivial abelian group with the only possible R-action: $r.0 = 0 \quad \forall r \in R$.

Finite direct sums (= products). – $M, N \in R\text{-mod}$

$M \oplus N = \{(m,n) : m \in M,\ n \in N\}$ with componentwise

$(= M \times N)$ operations: $\begin{cases} (m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2) \\ r.(m,n) = (r.m, r.n) \end{cases}$

[Recall: for mod-R, a right R module is again an abelian group M and ~~an~~ ~~opposite~~ ring hom $R^{op} \xrightarrow{\rho} \mathrm{End}_{gp}(M)$.

Scalar product in this case is written on the right:

$m \cdot r = \rho(r)(m)$. The only change is in (iii) of page 2:

$m.(r_1 r_2) = (m. r_1) \cdot r_2$ because $\rho(r_1 r_2) = \rho(r_2)\rho(r_1)$]

(8.1) Lemma. — In an additive category, finite direct sums and direct products are isomorphic.

Proof. Let $\mathcal{C}$ be an additive category and I be a set. Let $\{X_i\}_{i \in I}$ be a subset of the class of objects of $\mathcal{C}$. Assume $X := \bigoplus_{i \in I} X_i$ and $\hat{X} = \prod_{i \in I} X_i$ exist in $\mathcal{C}$. (We will impose finiteness of I, later when it is needed). Thus, we are given morphisms:

$$X_i \xrightarrow{f_i} \bigoplus_{j \in I} X_j = X \quad \text{and} \quad \hat{X} = \prod_{j \in I} X_j \xrightarrow{\hat{f}^i} X_i$$

$\forall\, i, j$; let $\delta_{ji} : X_i \to X_j$ be the morphism $\begin{cases} 0 & \text{if } i \neq j \\ \mathrm{Id}_{X_i} & \text{if } i = j \end{cases}$

Step 1. For a fixed $i \in I$, the collection of morphisms $\{ \delta_{ji} : X_i \to X_j \}_{\forall j \in I}$ define, by the universal property of direct product, a morphism $\hat{f}_i : X_i \longrightarrow \hat{X}$.

Similary the collection of morphisms $\{ \delta_{ij} : X_j \to X_i \}_{\forall j \in I}$ defines a morphism $\bigoplus_{j \in J} X_j = X \xrightarrow{f^i} X_i$

Because of the universal property we get

$$f^j f_i = \delta_{ji} \quad \text{and} \quad \hat{f}^j \hat{f}_i = \delta_{ji}.$$

[e.g. f^j is the unique map s.t. $\bigoplus_{k \in I} X_k = X \xrightarrow{f^j} X_j$

s.t. $\forall i \in I$

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] \arrow[dr, "\delta_{ji}"] & \\ X \arrow[r, "f^j"'] & X_j \end{tikzcd}$$

commutes.]

Step 2. $\alpha : \bigoplus_{i \in I} X_i = X \longrightarrow \hat{X} = \prod_{i \in I} X_i$ can be defined

either by • $\{X \xrightarrow{f^i} X_i\}_{\forall i \in I} \underset{}{\overset{\text{U.P. of } \prod}{\rightsquigarrow}} X \xrightarrow{\alpha} \hat{X}$

or by • $\{X_i \xrightarrow{\hat{f}_i} \hat{X}\}_{\forall i \in I} \underset{\text{U.P. of } \bigoplus}{\rightsquigarrow} X \xrightarrow{\alpha} \hat{X}$

Check.– In either case, $\alpha : X \to \hat{X}$ is the unique morphism s.t.

$$\begin{tikzcd} X_i \arrow[r, "\delta_{ji}"] \arrow[d, "f_i"'] & X_j \\ X \arrow[r, "\alpha"'] & \hat{X} \arrow[u, "\hat{f}^j"'] \end{tikzcd}$$

commutes $\forall i, j \in I$.

i.e. $\hat{f}^j \alpha f_i = \delta_{ji} \quad \forall i, j \in I.$

Step 3. Now assume I is finite and define

$\beta : \hat{X} \longrightarrow X$ as $\beta = \sum_{i \in I} f_i \hat{f}^i$

$\left(\hat{f}^i \downarrow X_i \nearrow^{f_i} \ \forall i \in I \right)$

$\llcorner$ sum in the abelian group $\mathrm{Hom}_{\mathcal{C}}(\hat{X}, X)$.

$$X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \hat{X}.$$

We will now show that $\beta \circ \alpha : X \to X$ is identity. I will leave the proof of $\alpha \circ \beta = \mathrm{Id}_{\hat{X}}$ which is exactly on the same lines.

$\beta\alpha : X \to X$. For every $i \in I$, we have

$$(\beta\alpha) \circ f_i = \sum_{j \in I} f_j \underbrace{\boxed{\hat{f}^j \alpha f_i}}_{= \delta_{ji}} = f_i$$

(see prev. page)

In the universal property of $\bigoplus_{j \in I} X_j$, set $Y = X$ & $(X_i \to Y) = f_i$ there must be a unique morphism $X \xrightarrow{g} Y$ $(\overset{\shortparallel}{X})$ s.t. $g \circ f_i = f_i \ \forall i \in I$.

$g = \mathrm{Id}_X$ and $g = \beta\alpha$ both satisfy this condition $\Rightarrow \mathrm{Id}_X = \beta\alpha$

$\square$

(8.2) Kernels and cokernels. ~~exist~~ - Again let $\mathcal{C}$ be an additive category and let $f : X \to Y$ be a morphism.

<u>Kernel of f</u> is a pair $(K, i : K \to X)$ such that

- i is injective
- $f \circ i = 0$
- for every $Z \in \mathcal{C}$ and $g : Z \to X$ s.t. $f \circ g = 0$, there is a (unique) $\overline{g} : Z \to K$ s.t. $g = i \circ \overline{g}$. Pictorially,

$$\begin{tikzcd}
K \arrow[r, "i"] & X \arrow[r, "f"] & Y \\
 & Z \arrow[ul, dashed, "\exists !\, \overline{g}"] \arrow[u, "g"] \arrow[ur, "=0"'] &
\end{tikzcd}$$

<u>Cokernel of f</u> is a pair $(C, \pi : Y \to C)$ s.t.

- π is surjective
- $\pi \circ f = 0$
- for every $Z \in \mathcal{C}$ and morphism $h : Y \to Z$ s.t. $h \circ f = 0$, there is a (unique) $\overline{h} : C \to Z$ s.t. $h = \overline{h} \circ \pi$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "0="'] & Y \arrow[r, "\pi"] \arrow[d, "h"] & C \arrow[dl, dashed, "\exists !\, \overline{h}"] \\
 & Z &
\end{tikzcd}$$

(8.2) Proposition. — Let $\mathcal{C}$ be an additive category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then

- f is injective iff $\mathrm{Ker}(f) = (0_{\mathcal{C}},\ 0_{\mathcal{C}} \xrightarrow{\text{unique}} X)$
- f is surjective iff $\mathrm{Coker}(f) = (0_{\mathcal{C}},\ Y \to 0_{\mathcal{C}})$
- Assume kernels and cokernels exist in $\mathcal{C}$. Define

$$\text{Image of } f = \mathrm{Im}(f) := \mathrm{Ker}(Y \xrightarrow[\mathrm{Coker}(f)]{\pi} C) \quad (\text{similarly } \mathrm{Coim}(f))$$

Then we have a ~~natural~~ commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ \mathrm{Coim}(f) \arrow[r, "\bar{f}"'] & \mathrm{Im}(f) \arrow[u, "i"'] \end{tikzcd}$$

i is injective
p is surjective
~~$\bar{f}$ is bijective.~~

Proof. — (1) f is injective $\Longleftrightarrow$ $\mathrm{Ker}(f) = 0$.

($\Rightarrow$) $0_{\mathcal{C}} \xrightarrow{0} X \xrightarrow{f} Y$ composition is 0 ✓

Let $Z \xrightarrow{g} X$ be a morphism s.t. $f \circ g = 0$. Then the same is true for

$$\begin{tikzcd} Z \arrow[rr, "0"] \arrow[dr] & & X \\ & 0_{\mathcal{C}} \arrow[ur] & \end{tikzcd}$$

As f is injective, we get $g = 0$.

i.e. g factors through $0_{\mathcal{C}} \to X$.

($\Leftarrow$) $\mathrm{Ker}(f)$ (exists) and is $0_{\mathcal{C}} \xrightarrow{0} X$. To prove: f is injective. Let $Z \overset{g_1}{\underset{g_2}{\rightrightarrows}} X$ be two morphisms s.t.

$f \circ g_1 = f \circ g_2$, i.e. $f \circ (g_1 - g_2) = 0$. By defn of kernel $g = g_1 - g_2$ must factor through

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr] & & X \\ & 0_{\mathcal{C}} \arrow[ur] & \end{tikzcd}$$

; i.e. $g = 0$

$\Rightarrow g_1 = g_2$.

(2) For the surjectivity case, use the same argument as above.

(3) Now let $X \xrightarrow{f} Y$ be any morphism in $\mathcal{C}$.

Let C be cokernel of f. That is, $X \xrightarrow{f} Y \xrightarrow{\pi} C$, $\pi \circ f = 0$.

By defn. of kernel $\exists\ X \longrightarrow \mathrm{Ker}(\pi) = \mathrm{Im}(f)$ s.t.

$$\begin{tikzcd} \mathrm{Im}(f) \arrow[r, "i"] & Y \arrow[r, "\pi"] & C \\ & X \arrow[u, "f"] \arrow[ur, "=0"'] \arrow[ul, dashed, "\tilde{f}"] & \end{tikzcd}$$

i is injective

$f = i \circ \tilde{f}$

Now let $\mathrm{Ker}(f) \xhookrightarrow{k} X$ be the kernel of f. Thus $f \circ k = 0$

$\Rightarrow i \circ \tilde{f} \circ k = 0 = i \circ (0) \Rightarrow \tilde{f} \circ k = 0$ by injectivity of i

This argument shows that $(\mathrm{Ker}(f), k)$ is also kernel of $\tilde{f}$ (the reader should fill in the details).

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "k"] \arrow[rr, bend right, "=0"'] & X \arrow[r, "\tilde{f}"] & \mathrm{Im}(f) \end{tikzcd}$$

. By defn. of Coimage (= cokernel of $\mathrm{Ker}(f) \xrightarrow{k} X$) we get

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "k"] \arrow[dr, "0"'] & X \arrow[r, "p"] \arrow[d, "\tilde{f}"] & \mathrm{CoIm}(f) \arrow[dl, dashed, "\bar{f}"] \\ & \mathrm{Im}(f) & \end{tikzcd}$$

s.t. $\bar{f} \circ p = \tilde{f}$

and p is surjective.

$\square$