Lecture 9

(9.0) Recall: an additive category is a category $\mathcal{C}$ where all $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are abelian groups so that composition is bilinear; moreover $\mathcal{C}$ has a zero object $0_{\mathcal{C}}$, and finite direct sums (hence products) exist in $\mathcal{C}$.

We introduced the notions of Kernel, Cokernel, image and coimage of a morphism in $\mathcal{C}$.

Remark. $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ can be defined in any category $\mathcal{A}$ that has a zero object $0_{\mathcal{A}}$. Having such an object immediately gives us a natural morphism

$$\begin{tikzcd} X \arrow[rr, "0"] \arrow[dr] & & Y \\ & 0_{\mathcal{A}} \arrow[ur] & \end{tikzcd}$$

(thus implying that $\mathrm{Hom}_{\mathcal{A}}(X,Y) \neq \phi$), and we define $\mathrm{Ker}(f)$ via exactly the same properties as in Defn (8.2) page 7.

Last time we also proved that any morphism $f: X \to Y$ in an additive category $\mathcal{C}$, which has kernels & cokernels of all its morphisms, factorizes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ (X/\mathrm{Ker} f) = \mathrm{Coim}(f) \arrow[r, "\bar{f}"] & \mathrm{Im}(f) \arrow[u, "i"'] \end{tikzcd}$$

i is injective

p is surjective

(9.1) Definition. — An additive category $\mathcal{A}$ is <u>abelian</u> if

(AB1) $\forall f : X \to Y$ in $\mathcal{A}$, Kernel & cokernel of f exist.

(AB2) $\forall f : X \to Y$, the induced morphism

$\bar{f} : \mathrm{Coim}(f) \left(= X/_{\mathrm{Ker}(f)} \right) \longrightarrow \mathrm{Im}(f)$ is an isomorphism.

Note. — By Prop. (8.3) page 8, if f is <u>bijective</u> in $\mathcal{A}$, an abelian category, then $f = \bar{f}$, is an <u>isomorphism.</u>

(9.2) The category R-mod is abelian (First iso. theorem).
Just for fun, let us check (AB1) and (AB2).

(AB1). — Let $f : M \longrightarrow N$ be hom. of R modules (left).
$K := \{ m \in M \mid f(m) = 0 \}$. $K \hookrightarrow M$ is the kernel of f. Because $\forall g : P \longrightarrow M$ s.t. $fg = 0$, we get that $g(x) \in K \ \forall x \in P \Rightarrow$

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[dr, "\exists"'] & M \\ & K \arrow[u, hook] \end{tikzcd}$$

Similarly $C := N /_{\text{submodule } \{ f(m) : m \in M \}}$.

(AB2.)
$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \\ M/\mathrm{Ker} f \arrow[r, "\bar{f}"] & \mathrm{Im}(f) \arrow[u, hook] \end{tikzcd}$$
To show: $\bar{f}$ is an iso.

(Exercise. – In R-mod, bijections are isomorphisms.)

The argument is standard. By defn., $\bar{f}(x+\mathrm{Ker} f) = f(x)$ thus $\bar{f}$ is surjective. $\bar{f}(x+\mathrm{Ker} f) = 0 \Rightarrow f(x)=0$ i.e. $x \in \mathrm{Ker}(f)$.

So $\bar{f}$ is injective ($\mathrm{Ker}(f)=0 \iff$ injective - Prop (8.3) p. 8).

(9.3) Let $\mathcal{Ab}^{(2)}$ = category of pairs of abelian groups $(G_1 \supset G_2)$ (i.e., G_2 is a subgroup of G_1)

$$\mathrm{Hom}_{\mathcal{Ab}^{(2)}}\left(\begin{matrix} G_1 \\ \cup \\ G_2 \end{matrix}, \begin{matrix} H_1 \\ \cup \\ H_2 \end{matrix} \right) = \left\{ \begin{matrix} \text{gp. hom } f: G_1 \to H_1 \text{ s.t.} \\ f(G_2) \subset H_2 \end{matrix} \right\}$$

Properties. – True / False.

(1) $\mathcal{Ab}^{(2)}$ has finite direct sums, Hom's are abelian groups w/ bilinear composition. $\mathcal{A}^{(2)}$ has a zero object, namely $\begin{pmatrix} \{0\} \\ \cup \\ \{0\} \end{pmatrix}$. Thus $\mathcal{A}^{(2)}$ is an additive category.

(2)
$$\operatorname{Ker}\left(\begin{array}{ccc} G_1 & \xrightarrow{f} & H_1 \\ \cup & & \cup \\ G_2 & \longrightarrow & H_2 \end{array}\right) = \begin{array}{c} \operatorname{Ker}(f) \\ \cup \\ \operatorname{Ker}(f|_{G_2}) \end{array}$$

(check)

$$\operatorname{Coker}\left(\begin{array}{ccc} H_1 & \xrightarrow{g} & G_1 \\ \cup & & \cup \\ H_2 & & G_2 \end{array}\right) = \begin{array}{c} G_1/g(H_1) \\ \cup \\ \text{Image of}\left(G_2/g(H_2) \longrightarrow G_1/g(H_2) \longrightarrow G_1/g(H_1) \right) \end{array}$$

$$G_1/g(H_1) = \frac{G/g(H_2)}{g(H_1)/g(H_2)}$$

e.g.
$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\mathrm{Id}} & \mathbb{Z} \\ \cup & & \cup \\ (0) & & 2\mathbb{Z} \end{array}$$
is bijective but does not have an inverse

Thus $\mathcal{Ab}^{(2)}$ is not an abelian category.

(9.4) Functors. Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories (or just additive categories). An <u>additive (covariant) functor</u> $F: \mathcal{C} \to \mathcal{D}$ is a functor ~~F~~ such that the map it induces

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{F} \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$$

$\forall\, X, Y \in \mathcal{C}$, is a group homomorphism.

(Similarly for contravariant functors.)

→ In the context of abelian (or additive categories) we only work with additive functors.

Lemma. — Let $F : \mathcal{C} \to \mathcal{D}$ be an additive (covariant) functor between two additive categories. Then,

(i) $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$

(ii) $F(X \oplus Y) = F(X) \oplus F(Y)$

Proof. (i) Note that for any additive category $\mathcal{A}$, $0_{\mathcal{A}}$ is the unique object A s.t. in $\mathrm{Hom}_{\mathcal{A}}(A, A)$, Id_A = zero morphism.

[($\Rightarrow$) clear from defn. ($\Leftarrow$) If $A \in \mathcal{A}$ is s.t. $\mathrm{Id}_A = 0$, we get: $\forall B \in \mathcal{A}$, $f \in \mathrm{Hom}_{\mathcal{A}}(B, A)$, $\underset{\substack{\uparrow \\ \text{defn. of Id}}}{f =} \mathrm{Id}_A \circ f = \underset{\substack{\uparrow \\ \text{composition is bilinear}}}{0 \circ f = 0}$

Illy $\forall g \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, $g = g \circ \mathrm{Id}_A = g \circ 0 = 0$.

Thus $\forall B \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(A, B)$ and $\mathrm{Hom}_{\mathcal{A}}(B, A)$ are trivial abelian groups $\Rightarrow A = 0_{\mathcal{A}}$]

Thus to prove (i) we only need to check that in $\mathrm{Hom}_{\mathcal{D}}(F(0_{\mathcal{C}}), F(0_{\mathcal{C}}))$, $\mathrm{Id}_{F(0_{\mathcal{C}})}$ = zero morphism.

But $\underset{\text{(true for any functor)}}{\mathrm{Id}_{F(0_{\mathcal{C}})} = F(\mathrm{Id}_{0_{\mathcal{C}}})} = \underset{\text{(as F is additive)}}{F(0) = 0}$ $\square$

(ii) Recall (from Lemma 8.1, p. 4): we have morphisms

$$\begin{tikzcd} X \arrow[dr, "f_1"] & & X \\ & X \oplus Y \arrow[ur, "f^1"] \arrow[dr, "f^2"'] & \\ Y \arrow[ur, "f_2"'] & & Y \end{tikzcd}$$

s.t. $f^i f_j = \delta_{ij}$ $\begin{pmatrix} \mathrm{Id} \text{ for } i=j \\ 0 \text{ for } i\neq j \end{pmatrix}$

& $f_1 f^1 + f_2 f^2 = \mathrm{Id}_{X\oplus Y}$

Applying F, we get

$$\begin{tikzcd} F(X) \arrow[dr, "F(f_1)"] & & F(X) \\ & F(X\oplus Y) \arrow[ur, "F(f^1)"] \arrow[dr, "F(f^2)"'] & \\ F(Y) \arrow[ur, "F(f_2)"'] & & F(Y) \end{tikzcd}$$

again with $\begin{cases} F(f^i)F(f_j) = \delta_{ij} \\ F(f_1)F(f^1) + F(f_2)F(f^2) = \mathrm{Id} \end{cases}$ (because F preserves Id, 0 and composition). We claim that this defines $F(X\oplus Y)$ as the direct sum of $F(X)$ and $F(Y)$ w/ structure morphisms $F(f_1)$ & $F(f_2)$.

We need to prove: $\forall\ Z \in \mathcal{D}$ and morphisms $g_1: F(X) \longrightarrow Z$, $g_2: F(Y) \to Z$,

$\exists!\ g: F(X\oplus Y) \to Z$ s.t. $g \circ F(f_i) = g_i$ $(i=1,2)$.

Define $g = g_1 F(f^1) + g_2 F(f^2)$

sum in $\mathrm{Hom}_{\mathcal{D}}(F(X\oplus Y), Z)$

The required eq^{n}s $(g \circ F(f_i) = g_i$ for $i=1,2)$ follow from $F(f^j)F(f_i) = \delta_{ji}$. To prove uniqueness of g, assume $\tilde{g}$ is another such morphism. $\left(\tilde{g}: F(X\oplus Y) \to Z \text{ s.t. } \tilde{g}F(f_i) = g_i \text{ for } i=1,2\right)$

We claim that $\tilde{g} = g$. This follows from

$$F(f_1)F(f^1) + F(f_2)F(f^2) = \mathrm{Id}_{F(X\oplus Y)} \cdot$$

$$\tilde{g} = \tilde{g}\circ \mathrm{Id}_{F(X\oplus Y)} = \tilde{g}\circ F(f_1)\circ F(f^1) + \tilde{g}\circ F(f_2)F(f^2)$$

$$= g_1 F(f^1) + g_2 F(f^2) \quad (\text{because } \tilde{g}F(f_i) = g_i)$$

$$= g \quad (\text{by defn.})$$

$\square$

(9.5) Exact functors. — If $\mathcal{C}$ is an abelian category, we say a pair of morphisms $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ is <u>exact</u> (or <u>exact at X_2</u>) if $\mathrm{Im}(f) = \mathrm{Ker}(g)$ (as usual)

A <u>short exact sequence in $\mathcal{C}$</u> is a sequence of morphisms

$$0_{\mathcal{C}} \xrightarrow{0} X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{0} 0_{\mathcal{C}}$$

exact at X_1, X_2 & X_3 (i.e. $\mathrm{Ker}(f) = 0 \Rightarrow f$ is injective $\left(\begin{smallmatrix}\text{Prop}\\ 8.3\\ \text{p. 8}\end{smallmatrix}\right)$

$\mathrm{Coker}(g) = 0 \Rightarrow g$ is surjective

and $\mathrm{Im}(f) = \mathrm{Ker}(g)$).

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an (covariant) additive functor between two abelian categories.

F is said to be <u>exact</u> if for every short exact sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ in $\mathcal{C}$,

$0 \to F(X_1) \to F(X_2) \to F(X_3) \to 0$ is exact in $\mathcal{D}$.

[<u>Left exact</u> : if $0 \to F(X_1) \to F(X_2) \to F(X_3)$ is exact in $\mathcal{D}$
<u>Right exact</u> : if $F(X_1) \to F(X_2) \to F(X_3) \to 0$ is exact in $\mathcal{D}$]

For contravariant $F : \mathcal{C} \to \mathcal{D}$, the same notions hold as:

$\forall\ 0 \to X_1 \to X_2 \to X_3 \to 0$ exact in $\mathcal{C}$

- $0 \to F(X_3) \to F(X_2) \to F(X_1) \to 0$ is exact in $\mathcal{D}$ (F is <u>exact</u>)
- $0 \to F(X_3) \to F(X_2) \to F(X_1)$ is exact in $\mathcal{D}$ (F is <u>left exact</u>)
- $F(X_3) \to F(X_2) \to F(X_1) \to 0$ is exact in $\mathcal{D}$ (F is <u>right exact</u>)

(9.6) Remark. $0 \to X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ exact $\iff$ (X_1, f) is the kernel of $X_2 \xrightarrow{g} X_3$

$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \to 0$ exact $\iff$ (X_3, g) is the cokernel of $X_1 \xrightarrow{f} X_2$.