

(10.0) Recall: for a category  $\mathcal{C}$ , we say

- $\mathcal{C}$  is additive if  $\text{Hom}$ 's are abelian groups; compositions are bilinear; and  $\mathcal{C}$  has zero object and finite direct sums (= finite direct products)
- $\mathcal{C}$  is abelian if it is additive and kernels & cokernels of morphisms exist; and finally  $\forall f: X \rightarrow Y$  a morphism in  $\mathcal{C}$ ,  $\bar{f}: \text{Coim}(f) (= X/\text{Ker}f) \rightarrow \text{Im}(f)$  is an isomorphism
- $F: \mathcal{C} \rightarrow \mathcal{D}$  a functor between two additive categories is an additive functor if  $\forall X, Y \in \mathcal{C}$ ,
 
$$\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F} \text{Hom}_{\mathcal{D}}(FX, FY)$$
 is a group homomorphism.

Last time we introduced the notion of exact/left exact/right exact functors between abelian categories.

(10.1) Thm. -  $\forall X \in \mathcal{C}$  (an abelian category), the (2)

Hom functors  $h_X = \text{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \rightarrow \text{Ab}$  (contra)

$h^X = \text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \text{Ab}$  (covariant)

are left exact.

Proof. - Let us prove it for the contravariant functor first. Let  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  be a short exact sequence in  $\mathcal{C}$ . We need to prove that:

$$0 \rightarrow h_X(C) \xrightarrow{h_X(g)} h_X(B) \xrightarrow{h_X(f)} h_X(A) \text{ is exact} \\ \parallel \qquad \qquad \qquad \parallel \qquad \qquad \qquad \parallel \text{ (of ab. gps.)}$$

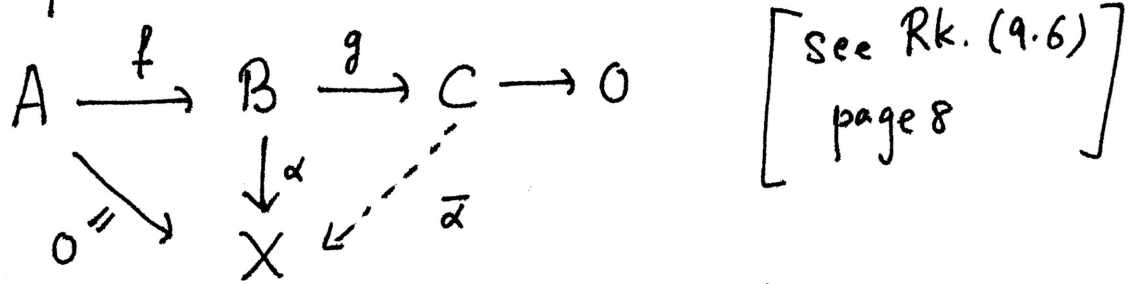
$$0 \rightarrow \text{Hom}(C, X) \xrightarrow{- \circ g} \text{Hom}(B, X) \xrightarrow{- \circ f} \text{Hom}(A, X)$$

Note. - injectivity of  $- \circ g$  is the definition of surjectivity of  $g$ .

As  $g \circ f = 0$ ,  $h_X(f) \circ h_X(g) = 0$ ,  $\Rightarrow$  image of  $h_X(g)$  is contained in kernel of  $h_X(f)$ . It remains to show

that  $\text{Ker}(h_X(f)) \subset \text{Im}(h_X(g))$ . So let  $\alpha: B \rightarrow X$

be such that  $\alpha \circ f = 0$ . As  $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  is exact, in particular,  $B \xrightarrow{g} C$  is the cokernel of  $f$ ; by defn. of cokernel,  $\exists!$   $\bar{\alpha}: C \rightarrow X$  s.t.  $\alpha = \bar{\alpha} \circ g$ .



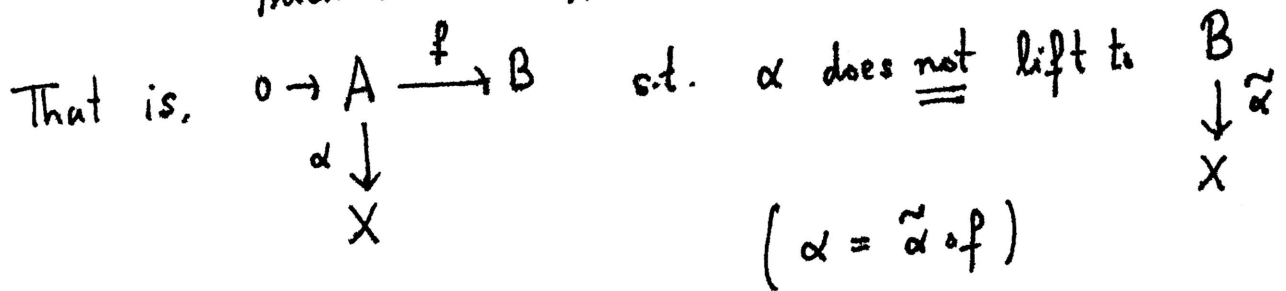
In other words,  $\text{Ker}(h_X(f)) \subset \text{Im}(h_X(g))$ . □

(The proof for the covariant functor is entirely analogous.)

Remark. - We didn't prove that  $h_X$  and  $h^X$  are additive functors, but I will leave that routine verification as an exercise.

(10.2) Important. - Hom's are not exact. Here are some standard counterexamples.

•  $h_X$ . Example of an injective morphism  $0 \rightarrow A \xrightarrow{f} B$  such that  $h_X(B) \rightarrow h_X(A)$  is not surjective.



Take  $\mathcal{C} = \text{Ab}$ ,  $A = B = X = \mathbb{Z}$ ,  $\alpha = \text{Id}$ ,  $f = \text{mult. by } 2$  (4)

$$\begin{array}{ccc}
 \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} \\
 \downarrow \alpha = \text{Id} & \swarrow \text{?} = \tilde{\alpha} & \\
 \mathbb{Z} & & 
 \end{array}$$

$\tilde{\alpha}(1) \in \mathbb{Z}$  should be some integer so that

2.  $\tilde{\alpha}(1) = 1$ . There is no such integer.

•  $h^X$ . Example of a surjective morphism  $B \xrightarrow{g} C \rightarrow 0$   
 s.t.  $h^X(B) \rightarrow h^X(C) \rightarrow 0$  (not exact).  
 $\text{Hom}(X, B) \rightarrow \text{Hom}(X, C)$   
 $g_*$

That is, example of  $B \rightarrow C \rightarrow 0$  s.t.  $\alpha$  does not lift to  $B$   
 $\uparrow \tilde{\alpha}$   
 $X$

Take  $\mathcal{C} = \text{Ab}$ .  $\mathbb{Z} \xrightarrow{g = \text{projection}} \mathbb{Z}/2\mathbb{Z} \rightarrow 0$   
 $\uparrow \text{Id} = \alpha$   
 $\mathbb{Z}/2\mathbb{Z}$

But  $\text{Hom}_{\text{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = (0)$ . So there is no such  $\tilde{\alpha}$ .

This is a feature of Hom functors, not a bug.

(10.3) Definition. Let  $\mathcal{A}$  be an abelian category. An object  $X \in \mathcal{A}$  is said to be projective (resp. injective) if  $\text{Hom}_{\mathcal{A}}(X, -)$  is exact (resp. if  $\text{Hom}_{\mathcal{A}}(-, X)$  is exact). (5)

(10.4) We already know that  $\{X_i\}_{i \in I} \longmapsto \bigoplus_{i \in I} X_i$  is right exact (covariant) functor (of course, if it exists!)

For  $|I| < \infty$ ,  $\mathcal{C}^I \longrightarrow \mathcal{C}$  ( $\mathcal{C}$ : an abelian category)

$$\{X_i\} \longmapsto \bigoplus_{i \in I} X_i \\ (= \prod_{i \in I} X_i)$$

is an exact functor.

In HW4, you will have to prove that for  $\mathcal{A} = R\text{-mod}$

$$\bigoplus_{i \in I} M_i, \quad \prod_{i \in I} M_i \text{ exist and are exact.}$$

$\varinjlim (\{M_i\}_{i \in I}; \{\psi_{ji}: M_i \rightarrow M_j\}_{i \leq j})$  exists and

is exact. Inverse limit exists but is not exact (only left exact.)