Lecture 10

(10.0) Recall : for a category $\mathcal{C}$, we say

- $\mathcal{C}$ is additive if Hom's are abelian groups; compositions are bilinear; and $\mathcal{C}$ has zero object and finite direct sums (= finite direct products)
- $\mathcal{C}$ is abelian if it is additive and kernels & cokernels of morphisms exist; and finally $\forall f : X \to Y$ a morphism in $\mathcal{C}$, $\bar{f} : \mathrm{Coim}(f) \left(= X/_{\mathrm{Ker} f}\right) \longrightarrow \mathrm{Im}(f)$ is an isomorphism
- $F : \mathcal{C} \to \mathcal{D}$ a functor between two additive categories is an additive functor if $\forall X, Y \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(FX, FY)$$

is a group homomorphism.

Last time we introduced the notion of exact / left exact / right exact functors between abelian categories.

(10.1) Thm. – $\forall X \in \mathcal{C}$ (an abelian category), the Hom functors
$$h_X = \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \longrightarrow \mathcal{Ab} \text{ (contra)}$$
$$h^X = \mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathcal{Ab} \text{ (covariant)}$$
are left exact.

Proof. - Let us prove it for the contravariant functor first. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\mathcal{C}$. We need to prove that

$$\begin{array}{ccccccc} 0 \longrightarrow & h_X(C) & \xrightarrow{h_X(g)} & h_X(B) & \xrightarrow{h_X(f)} & h_X(A) & \text{is exact} \\ & \| & & \| & & \| & \text{(of ab. gps.)} \\ 0 \to & \mathrm{Hom}(C,X) & \xrightarrow[-\circ g]{} & \mathrm{Hom}(B,X) & \xrightarrow[-\circ f]{} & \mathrm{Hom}(A,X) & \end{array}$$

Note. - injectivity of $-\circ g$ is the definition of surjectivity of g.

As $g \circ f = 0$, $h_X(f) \circ h_X(g) = 0$, $\Rightarrow$ image of $h_X(g)$ is contained in Kernel of $h_X(f)$. It remains to show that $\mathrm{Ker}(h_X(f)) \subset \mathrm{Im}(h_X(g))$. So let $\alpha : B \to X$

be such that $\alpha \circ f = 0$. As $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, in particular, $B \xrightarrow{g} C$ is the cokernel of f; by defn. of cokernel, $\exists!\ \bar{\alpha} : C \to X$ s.t. $\alpha = \bar{\alpha} \circ g$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0 ="'] & B \arrow[r, "g"] \arrow[d, "\alpha"] & C \arrow[r] \arrow[dl, dashed, "\bar{\alpha}"] & 0 \\ & X & & \end{tikzcd}$$

[See Rk. (9.6) page 8]

In other words, $\mathrm{Ker}(h_X(f)) \subset \mathrm{Im}(h_X(g))$. $\square$

(The proof for the covariant functor is entirely analogous.)

Remark. - We didn't prove that h_X and h^X are additive functors, but I will leave that routine verification as an exercise.

(10.2) Important. - Hom's are <u>not</u> exact. Here are some standard counterexamples.

- h_X. Example of an injective morphism $0 \to A \xrightarrow{f} B$ such that $h_X(B) \longrightarrow h_X(A)$ is <u>not</u> surjective.

That is,
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \\ & X & \end{tikzcd}$$
s.t. α does <u>not</u> lift to
$$\begin{tikzcd} B \arrow[d, "\tilde{\alpha}"] \\ X \end{tikzcd}$$

$(\alpha = \tilde{\alpha} \circ f)$

Take $\mathcal{C} = \mathcal{Ab}$, $A = B = X = \mathbb{Z}$, $\alpha = \mathrm{Id}$, $f = $ mult. by 2

$$\begin{tikzcd} & x \arrow[r, mapsto] & 2x \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "f"] \arrow[d, "\alpha = \mathrm{Id}"'] & \mathbb{Z} \arrow[dl, dashed, "? = \tilde{\alpha}"] \\ & \mathbb{Z} & \end{tikzcd}$$

$\tilde{\alpha}(1) \in \mathbb{Z}$ should be some integer so that $2 \cdot \tilde{\alpha}(1) = 1$. There is no such integer.

- h^X. Example of a surjective morphism $B \xrightarrow{g} C \to 0$ s.t. $h^X(B) \longrightarrow h^X(C) \to 0$ (<u>not</u> exact).

$$\begin{array}{ccc} h^X(B) & \longrightarrow & h^X(C) \\ \| & & \| \\ \mathrm{Hom}(X,B) & \xrightarrow[g\circ -]{} & \mathrm{Hom}(X,C) \end{array}$$

That is, example of

$$\begin{tikzcd} B \arrow[r] & C \arrow[r] & 0 \\ & X \arrow[u, "\alpha"] \arrow[ul, dashed, "\tilde{\alpha} = ?"] & \end{tikzcd}$$

s.t. α does not lift to $\begin{matrix} B \\ \uparrow \tilde{\alpha} \\ X \end{matrix}$

Take $\mathcal{C} = \mathcal{Ab}$.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "g = \text{projection}"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \\ & \mathbb{Z}/2\mathbb{Z} \arrow[u, "\mathrm{Id} = \alpha"'] \arrow[ul, dashed, "? = \tilde{\alpha}"] & \end{tikzcd}$$

But $\mathrm{Hom}_{\mathcal{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = (0)$. So there is no such $\tilde{\alpha}$.

This is a <u>feature</u> of Hom functors, not a <u>bug</u>.

(10.3) Definition. Let $\mathcal{A}$ be an abelian category. An object $X \in \mathcal{A}$ is said to be <u>projective</u> (resp. <u>injective</u>) if $\mathrm{Hom}_{\mathcal{A}}(X, -)$ is exact (resp. if $\mathrm{Hom}_{\mathcal{A}}(-, X)$ is exact).

(10.4) We already know that $\{X_i\}_{i\in I} \longmapsto \bigoplus_{i\in I} X_i$ is right exact (covariant) functor (of course, if it exists!)

For $|I| < \infty$, $\mathcal{C}^I \longrightarrow \mathcal{C}$ ($\mathcal{C}$: an abelian category)

$$\{X_i\} \longmapsto \bigoplus_{i\in I} X_i \quad \left(= \prod_{i\in I} X_i\right)$$

is an exact functor.

In HW4, you will have to prove that for $\mathcal{A} = R\text{-mod}$

$\bigoplus_{i\in I} M_i$, $\prod_{i\in I} M_i$ exist and are exact.

$\varinjlim \left(\{M_i\}_{i\in I} ; \{\varphi_{ji} : M_i \to M_j\}_{i\le j}\right)$ exists and is exact. Inverse limit exists but is <u>not</u> exact (only left exact.)