

(11.0) The big picture. - Our aim for the next few weeks is to define and compute Ext and Tor functors.

As we saw last time, $h^X = \text{Hom}_A(X, -)$ and $h_X = \text{Hom}_A(-, X)$ (for an abelian category A and $X \in A$) are both left exact functors from A to Ab (category of abelian groups).

The failure of these functors to be exact is measured in terms of $\text{Ext}_A^i(X, -)$ and $\text{Ext}_A^i(-, X)$ - "derived functors of h^X & h_X respectively." (see defn. below).

This week we will also define a bifunctor $-\otimes-$. This bifunctor is not defined for an arbitrary abelian category, but only for module categories. (Its abstraction via representability of functors exist - called multicategories - but is beyond the scope of this course).

As a functor $\otimes : \text{mod-}R \times R\text{-mod} \longrightarrow \text{Ab}$
 $(M, N) \longmapsto M \otimes_R N.$

We will focus mainly on commutative rings, in which case $(R\text{-mod} = \text{mod-}R)$ and there is a natural structure of R -module on $M \otimes_R N$: ②

$$\otimes : R\text{-mod} \times R\text{-mod} \longrightarrow R\text{-mod.} \quad [R: \text{comm.}]$$

For a fixed $X \in R\text{-mod}$, $X \otimes -$ (actually same as $- \otimes X$) is (covariant) right-exact functor. Its "derived functors" are denoted by $\text{Tor}_R^i(X, -)$.

[Similarly, for h^X and h_X , if $X \in \mathcal{A} = R\text{-mod}$, where R is a commutative ring, (with $1 \neq 0$), h^X and h_X map $R\text{-mod}$ to $R\text{-mod}$, and hence so do their derived functors.]

(11.1) Derived functors of left exact functors.

Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive, left exact (covariant or contravariant) functor between two abelian categories \mathcal{A} and \mathcal{B} . (Right) derived functors of F , denoted by $R^i F$, are functors

$$(\forall i \geq 0) \quad R^i F : \mathcal{A} \longrightarrow \mathcal{B}$$

together with the data of

(1) ^{natural} An isomorphism of functors $F \simeq R^0 F$

(2) Connecting homomorphisms: $\forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ exact in A , a morphism $R^i F(Z) \xrightarrow{\partial} R^{i+1} F(X)$ covariant case
 $[R^i F(X) \xrightarrow{\partial} R^{i+1} F(Z)]$ contravariant case

subject to following conditions.

(1) $\forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ exact in A , the following sequence is exact in B .

$$0 \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow R^i F(X) \rightarrow \dots \rightarrow R^i F(Z) \rightarrow R^{i+1} F(X) \rightarrow \dots$$

(covariant case)

$$[0 \rightarrow F(Z) \rightarrow F(Y) \rightarrow F(X) \rightarrow R^i F(Z) \rightarrow \dots]$$

(contra case)

(2) For every commutative diagram of short exact seq. in A :

$$\begin{array}{ccccccc} 0 & \rightarrow & X_1 & \rightarrow & Y_1 & \rightarrow & Z_1 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & X_2 & \rightarrow & Y_2 & \rightarrow & Z_2 \rightarrow 0 \end{array}$$

We get a commutative "ladder" of exact sequences in B

$$\begin{array}{ccc} \dots & \rightarrow & R^i F(Z_1) \rightarrow R^{i+1} F(X_1) \rightarrow \dots \\ & & \downarrow \qquad \qquad \downarrow \\ \dots & \rightarrow & R^i F(Z_2) \rightarrow R^{i+1} F(X_2) \rightarrow \dots \end{array}$$

covariant case

$$\left[\begin{array}{ccc} \dots & \rightarrow & R^i F(X_1) \rightarrow R^{i+1} F(Z_1) \rightarrow \dots \\ & \uparrow & \uparrow \\ \dots & \rightarrow & R^i F(X_2) \rightarrow R^{i+1} F(Z_2) \rightarrow \dots \end{array} \right]$$

contravariant case

(11.2) Derived functors of right exact functors.

Let $G: A \rightarrow B$ now be a right exact functor (for convenience, restrict to covariant case). Derived functors of G , denoted by $L_i G$, are functors (additive, as usual).

$\forall i \geq 0$, $L_i G: A \rightarrow B$; together with

(1) An isomorphism (natural) of functors $G \simeq L_0 G$.

(2) Connecting morphisms: $\forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in A (exact)

$$L_k^G(Z) \rightarrow L_{k-1}^G(X)$$

such that (1) $\dots \rightarrow L_k G(Y) \rightarrow L_{k-1} G(Z) \rightarrow G(X) \rightarrow G(Y) \rightarrow G(Z) \rightarrow 0$ is exact in B .

(2) for every commutative diagram of short exact sequences in

$$\begin{array}{ccccccc}
A: & 0 & \rightarrow & X_1 & \rightarrow & Y_1 & \rightarrow & Z_1 & \rightarrow & 0 \\
& & & \downarrow & & \downarrow & & \downarrow & & \\
& 0 & \rightarrow & X_2 & \rightarrow & Y_2 & \rightarrow & Z_2 & \rightarrow & 0
\end{array}$$

we get a

commutative "ladder" in B :

$$\begin{array}{ccccccc}
\dots & \rightarrow & L_k G(Z_1) & \rightarrow & L_{k-1} G(X_1) & \rightarrow & \dots \\
& & \downarrow & & \downarrow & & \\
\dots & \rightarrow & L_k G(Z_2) & \rightarrow & L_{k-1} G(X_2) & \rightarrow & \dots
\end{array}$$

(11.3) Some remarks and heuristical arguments. -

for fixing ideas

Starting from $F: A \rightarrow B$ (say covariant, left exact) one can begin a search for "universal functors" $\{S^i F\}_{i \geq 0}$ satisfying the conditions imposed by definition (11.1). Such generalizations exist (under the name of satellite functors) but we will only address them in Optional Reading B.

For now, let us investigate what properties of A (or B) will make sure that $R^1 F$ can be defined (hopefully unambiguously).

So let $X \in A$, and see how Defn (11.1) would constrain $R^1 F(X)$. Our only guiding principle is:

$$\begin{array}{ccc}
 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 & \text{and} & 0 \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow R^1 F(X) \\
 \text{exact in } A & & \rightarrow R^1 F(Y) \rightarrow \dots
 \end{array}$$

Observe: if $R^1 F(Y) = 0$, then $R^1 F(X) = \text{Coker}(F(Y) \rightarrow F(Z))$ (has to be!)

So the rule is clear (although a bit circular at the moment):

Given $X \in A$, find an injective morphism $X \rightarrow Y$ where

(6)

$R^1F(Y)$ should be zero (this sounds circular and we return to this point in a moment). Then define (& hope for the best!)

$$R^1F(X) = \text{Coker}(F(Y) \rightarrow F(Y/X))$$

Now what does $R^1F(Y) = 0$ mean since we are actually looking for R^1F ? Again our guiding principle says

" $R^1F(Y) = 0$ " means $\forall 0 \rightarrow Y \rightarrow \tilde{Y} \rightarrow \tilde{Y}/Y \rightarrow 0$ exact in \mathcal{A}
 $\Rightarrow 0 \rightarrow F(Y) \rightarrow F(\tilde{Y}) \rightarrow F(\tilde{Y}/Y) \rightarrow 0$ exact in \mathcal{B} .

So we arrive at a more precise "algorithm".

Input F

left exact,
covariant

\rightsquigarrow Define: F -acyclic objects of \mathcal{A} as $Y \in \mathcal{A}$ s.t. (or exact)

$$\forall 0 \rightarrow Y \rightarrow \tilde{Y} \rightarrow \tilde{Y}/Y \rightarrow 0 \text{ exact in } \mathcal{A}$$

$$0 \rightarrow F(Y) \rightarrow F(\tilde{Y}) \rightarrow F(\tilde{Y}/Y) \rightarrow 0 \text{ is exact in } \mathcal{B}$$

Hope: every $X \in \mathcal{A}$ admits an injective map $0 \rightarrow X \rightarrow Y$ for some F -acyclic Y .

Output $R^1F(X) := \text{Coker}(F(Y) \rightarrow F(Y/X))$

Prove: independence from $X \hookrightarrow Y$ chosen.

Once we understand R^1F this way, the higher derived functors are obtained recursively. (This is an instructive exercise that we arrive at the following answer).

Assume \mathcal{A} has "enough F -exact objects" (i.e. $\forall X \in \mathcal{A}$, \exists an injective morphism $X \rightarrow Y$ where Y is F -exact).

• For $X \in \mathcal{A}$, build an exact sequence

$$0 \rightarrow X \rightarrow Y_0^0 \rightarrow Y_1^1 \rightarrow Y_2^2 \rightarrow \dots$$

when each Y_j^j ($j \geq 0$) is F -exact.

(Y^0 exists by assumption, $X \xrightarrow{i} Y^0 \rightarrow Y^0/X \rightarrow Y^1 \dots$ exists by assumption)

• Apply F to get a "complex"

$$0 \rightarrow F(Y^0) \rightarrow F(Y^1) \rightarrow \dots$$

$$\bullet \quad R^k F(Y) := \frac{\text{Ker} (F(Y^k) \rightarrow F(Y^{k+1}))}{\text{Im} (F(Y^{k-1}) \rightarrow F(Y^k))}$$

This would leave us to prove that the answer does not depend on the choice made in the first step (called resolution of X - by F -exact objects).

Also what are connecting homomorphisms $R^k F(Z) \rightarrow R^{k+1} F(X)$?

Thus we get a "to-do list" for each $F: \mathcal{A} \rightarrow \mathcal{B}$ ⑧
(see (11.6) & (11.7) below) (covariant, left-exact)

(11.4) What are "F-exact" objects for $F = \text{Hom}_{\mathcal{A}}(X, -) = h^X$?

Answer. - Injective objects are F-exact for any F !! ← covariant

Theorem. For $Q \in \mathcal{A}$, the following are equivalent.

(1) Q is injective (2) \forall injective morphism $Q \hookrightarrow \tilde{Q}$

\tilde{Q} is iso. to $Q \oplus \bar{Q}$

(3) Q is F-exact \forall F additive functor (covariant).

(11.5) What are F^G -exact objects for $G = \text{Hom}_{\mathcal{A}}(-, X) = h_X$?

Answer. - Projective objects are G-exact for any contra G!!

Thm. $P \in \mathcal{A}$ is projective $\Leftrightarrow \forall$ surjection $\tilde{P} \rightarrow P$,

we have a direct sum decomposition $\tilde{P} \simeq P \oplus \bar{P}$.

Remark. - In concrete situations, there are usually many more

"F-exact" objects, which can be used as a replacement

for inj/proj objects, if inj/proj do not exist.

(11.6) Big picture. - How to "compute" derived functors? ⑨

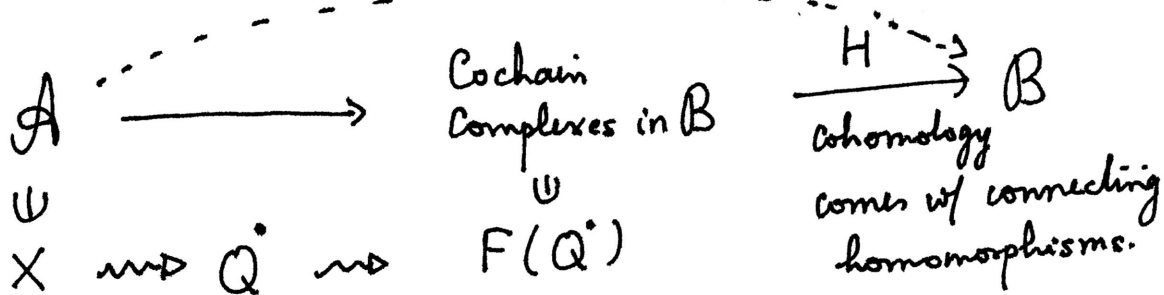
• $F : A \rightarrow B$ left exact covariant.

$X \in A$. Step 1. - replace X by its injective resolution
 $X \rightsquigarrow Q^\bullet$

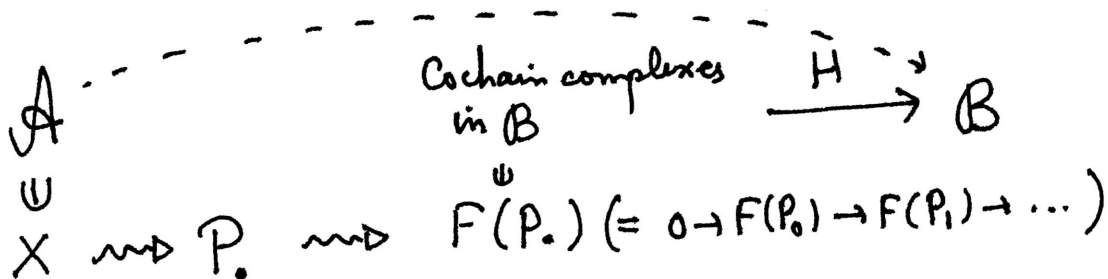
Step 2. - apply F to Q^\bullet (cochain)
 $0 \rightarrow F(Q^0) \rightarrow F(Q^1) \rightarrow \dots$ a complex in B

Step 3. - Take cohomology

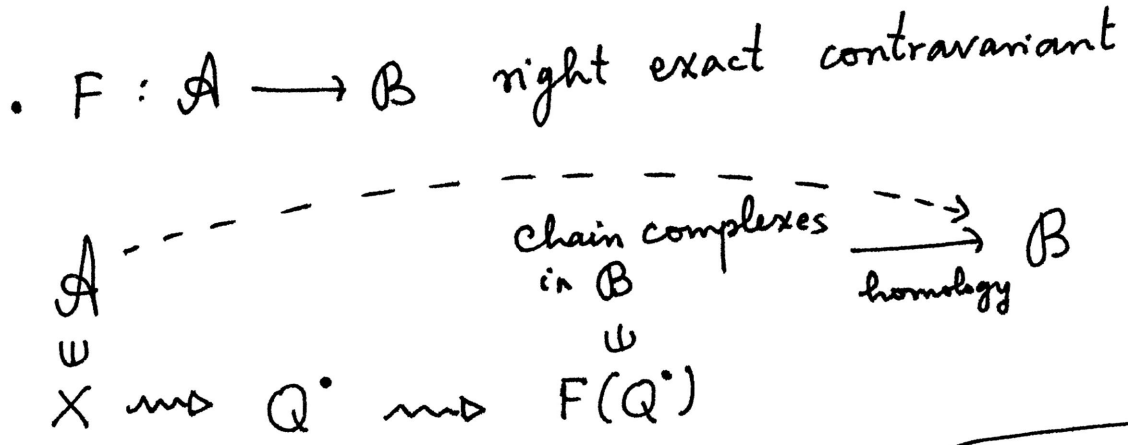
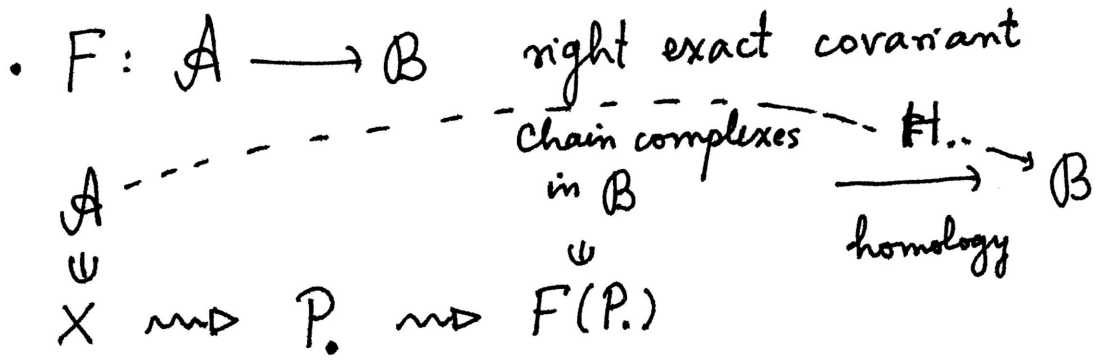
$$R^k F(X) = \frac{\text{Ker}(F(Q^k) \rightarrow F(Q^{k+1}))}{\text{Im}(F(Q^{k-1}) \rightarrow F(Q^k))}$$



• $F : A \rightarrow B$ left exact contravariant



(proj. res. $\dots \rightarrow P_k \rightarrow \dots \rightarrow P_0 \rightarrow X \rightarrow 0$)
 exact in A



$$Z_k F(X) = \frac{\text{Ker}(F(P_k) \rightarrow F(P_{k-1}))}{\text{Im}(F(P_{k+1}) \rightarrow F(P_k))}$$

(11.7) To do list. -

1. Understand how connecting homomorphisms are defined.

for $H_{(or.)}^i: \text{Complexes over } \mathcal{B} \rightarrow \mathcal{B}$

2. Make sure we have enough injective / projective objects.

3. Make sure the dotted lines from (11.6) above do not depend on the choice of P or Q .