# Lecture 11



(11.0) The big picture.– Our aim for the next few weeks is to define and compute Ext and Tor functors.

As we saw last time, $h^X = \mathrm{Hom}_{\mathcal{A}}(X,-)$ and $h_X = \mathrm{Hom}_{\mathcal{A}}(-,X)$ (for an abelian category $\mathcal{A}$ and $X \in \mathcal{A}$) are both left exact functors from $\mathcal{A}$ to $\mathcal{Ab}$ (category of abelian groups).

The failure of these functors to be exact is measured in terms of $\mathrm{Ext}^i_{\mathcal{A}}(X,-)$ and $\mathrm{Ext}^i_{\mathcal{A}}(-,X)$ – "derived functors of $h^X$ & $h_X$ respectively." (see defn. below).

This week we will also define a bifunctor $- \otimes -$. This bifunctor is not defined for an arbitrary abelian category, but only for module categories. (Its abstractions via representability of functors exist – called multicategories – but is beyond the scope of this course).

As a functor
$$\otimes : \text{mod-}R \times R\text{-mod} \longrightarrow \mathcal{Ab}$$
$$(M,N) \longmapsto M \otimes_R N.$$

We will focus mainly on commutative rings, in which case (R-mod = mod-R) and there is a natural structure of R-module on $M \otimes_R N$:

$$\otimes : R\text{-mod} \times R\text{-mod} \longrightarrow R\text{-mod} \quad [R: \text{comm.}]$$

For a fixed $X \in R$-mod, $X \otimes -$ (actually same as $- \otimes X$) is (covariant) right-exact functor. Its "derived functors" are denoted by $\mathrm{Tor}^i_R(X,-)$.

[ Similarly, for $h^X$ and $h_X$, if $X \in \mathcal{A} = R$-mod, where $R$ is a commutative ring, (with $1 \neq 0$), $h^X$ and $h_X$ map $R$-mod to $R$-mod, and hence so do their derived functors.]

(11.1) Derived functors of left exact functors.

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive, left exact (covariant or contravariant) functor between two abelian categories $\mathcal{A}$ and $\mathcal{B}$. (Right) derived functors of $F$, denoted by $R^iF$, are functions

$(\forall\, i \geq 0) \qquad R^iF : \mathcal{A} \longrightarrow \mathcal{B}$

together with the data of

(1) A natural isomorphism of functors $F \xrightarrow{\sim} R^0 F$

(2) Connecting homomorphisms: $\forall\ 0 \to X \to Y \to Z \to 0$ exact in $\mathcal{A}$, a morphism $R^i F(Z) \xrightarrow{\partial} R^{i+1} F(X)$ covariant case

$\left[ R^i F(X) \xrightarrow{\partial} R^{i+1} F(Z) \right.$ contravariant case $\left. \right]$

subject to following conditions.

(1) $\forall\, 0 \to X \to Y \to Z \to 0$ exact in $\mathcal{A}$, the following sequence is exact in $\mathcal{B}$.

$$\underset{(= R^0F(X))}{0 \to F(X)} \to F(Y) \to F(Z) \to R^1F(X) \to \cdots \to R^iF(Z) \to R^{i+1}F(X) \to \cdots$$

(covariant case)

$\left[ 0 \to F(Z) \to F(Y) \to F(X) \to R^1F(Z) \to \cdots \right.$ (contra case) $\left. \right]$

(2) For every commutative diagram of short exact seq. in $\mathcal{A}$:

$$\begin{tikzcd} 0 \arrow[r] & X_1 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X_2 \arrow[r] & Y_2 \arrow[r] & Z_2 \arrow[r] & 0 \end{tikzcd}$$

We get a commutative "ladder" of exact sequences in $\mathcal{B}$

$$\begin{tikzcd} \cdots \arrow[r] & R^iF(Z_1) \arrow[r] \arrow[d] & R^{i+1}F(X_1) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & R^iF(Z_2) \arrow[r] & R^{i+1}F(X_2) \arrow[r] & \cdots \end{tikzcd}$$

covariant case

$$\left[ \begin{tikzcd} \cdots \arrow[r] & R^iF(X_1) \arrow[r] & R^{i+1}F(Z_1) \arrow[r] & \cdots \\ \cdots \arrow[r] & R^iF(X_2) \arrow[r] \arrow[u] & R^{i+1}F(Z_2) \arrow[r] \arrow[u] & \cdots \end{tikzcd} \right]$$

contravariant case

## (11.2) Derived functors of right exact functors.

Let $G : \mathcal{A} \to \mathcal{B}$ now be a right exact functor (for convenience, restrict to covariant case). Derived functors of $G$, denoted by $L_i G$, are functors (additive, as usual).

$$\forall\ i \geq 0\ ,\quad L_i G : \mathcal{A} \longrightarrow \mathcal{B}\ ;\ \text{together with}$$

(1) An isomorphism (natural) of functors $G \simeq L_0 G$.

(2) Connecting morphisms : $\forall\ 0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ (exact)

$$L_k G(Z) \longrightarrow L_{k-1} G(X)$$

such that (1)

$$\cdots \to L_1 G(Y) \to L_1 G(Z) \to G(X) \to G(Y) \to G(Z) \to 0$$

is exact in $\mathcal{B}$.

(2) for every commutative diagram of short exact sequences in

$\mathcal{A}$ ;
$$\begin{array}{ccccccccc} 0 & \to & X_1 & \to & Y_1 & \to & Z_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X_2 & \to & Y_2 & \to & Z_2 & \to & 0 \end{array}$$
, we get a

commutative "ladder" in $\mathcal{B}$ :

$$\begin{array}{ccccccc} \cdots & \to & L_k G(Z_1) & \longrightarrow & L_{k-1} G(X_1) & \to & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \to & L_k G(Z_2) & \longrightarrow & L_{k-1} G(X_2) & \to & \cdots \end{array}$$

(11.3) Some remarks and heuristical arguments.-

Starting from $F : \mathcal{A} \to \mathcal{B}$ (say covariant, left exact) ↙ for fixing ideas one can begin a search for "universal functors" $\{S^i F\}_{i \geq 0}$ satisfying the conditions imposed by definition (11.1). Such generalizations exist (under the name of satellite functors) but we will only address them in Optional Reading B.

For now, let us investigate what properties of $\mathcal{A}$ (or $\mathcal{B}$) will make sure that $R^1 F$ can be defined (hopefully unambiguously).

So let $X \in \mathcal{A}$, and see how Defn (11.1) would constrain $R^1 F(X)$. Our only guiding principle is:

$$0 \to X \to Y \to Z \to 0 \quad \rightsquigarrow \quad 0 \to F(X) \to F(Y) \to F(Z) \to R^1F(X) \to R^1F(Y) \to \cdots$$

exact in $\mathcal{A}$

Observe: if $R^1F(Y) = 0$, then $R^1F(X) = \operatorname{Coker}(F(Y) \to F(Z))$ (has to be!)

So the rule is clear (although a bit circular at the moment):

Given $X \in \mathcal{A}$, find an injective morphism $X \to Y$ where

$R^1F(Y)$ should be zero (this sounds circular and we return to this point in a moment). Then define ($\&$ hope for the best!)

$$R^1F(X) = \operatorname{Coker}\left(F(Y) \longrightarrow F(Y/X)\right)$$

Now what does $R^1F(Y)=0$ mean since we are actually looking for $R^1F$? Again our guiding principle says

"$R^1F(Y)=0$" means $\forall\ 0 \to Y \to \tilde{Y} \to \tilde{Y}/Y \to 0$ exact in $\mathcal{A}$

$\Rightarrow\ 0 \to F(Y) \to F(\tilde{Y}) \to F(\tilde{Y}/Y) \to 0$ exact in $\mathcal{B}$.

So we arrive at a more precise "algorithm".

$\boxed{\text{Input } F}$ $\rightsquigarrow$ Define: $F$-acyclic (or exact) objects of $\mathcal{A}$ as $Y \in \mathcal{A}$ s.t.

left exact, covariant

$\forall\ 0 \to Y \to \tilde{Y} \to \tilde{Y}/Y \to 0$ exact in $\mathcal{A}$

$0 \to F(Y) \to F(\tilde{Y}) \to F(\tilde{Y}/Y) \to 0$ is exact in $\mathcal{B}$

Hope: every $X \in \mathcal{A}$ admits an injective map $0 \to X \to Y$ for some $F$-acyclic $Y$.

$$\boxed{\text{Output } R^1F(X) := \operatorname{Coker}\left(F(Y) \to F(Y/X)\right)}$$

Prove: independence from $X \hookrightarrow Y$ chosen.

Once we understand $R^1F$ this way, the higher derived functors are obtained recursively. (This is an instructive exercise that we arrive at the following answer).

Assume $\mathcal{A}$ has "enough F-exact objects" (i.e. $\forall X \in \mathcal{A}$, $\exists$ an injective morphism $X \to Y$ where $Y$ is F-exact).

- For $X \in \mathcal{A}$, build an exact sequence
$$0 \to X \to Y^0 \to Y^1 \to Y^2 \to \cdots$$
where each $Y^j$ ($j \geq 0$) is F-exact.

( $Y^0$ exists by assumption,
$$\begin{tikzcd} X \arrow[r, "i"] & Y^0 \arrow[rr] \arrow[dr] & & Y^1 & \cdots \\ & & Y^0/X \arrow[ur] & & \end{tikzcd}$$
$Y^1$ exists by assumption )

- Apply F to get a "complex"
$$0 \to F(Y^0) \to F(Y^1) \to \cdots$$

- $$R^kF(Y) := \frac{\operatorname{Ker}\left(F(Y^k) \to F(Y^{k+1})\right)}{\operatorname{Im}\left(F(Y^{k-1}) \to F(Y^k)\right)}$$

This would leave us to prove that the answer does not depend on the choice made in the first step (called resolution of X - by F-exact objects).

Also what are connecting homomorphisms $R^kF(Z) \to R^{k+1}F(X)$?

Thus we get a "to-do list" for each $F: \mathcal{A} \to \mathcal{B}$ (covariant, left-exact)
(see (11.6) & (11.7) below)

(11.4) What are "$F$-exact" objects for $F = \mathrm{Hom}_{\mathcal{A}}(X,-) = h^X$ ?

Answer. – Injective objects are $F$-exact for any $F$ !! ← covariant

Theorem. For $Q \in \mathcal{A}$, the following are equivalent.

(1) $Q$ is injective (2) $\forall$ injective morphism $Q \hookrightarrow \tilde{Q}$

$\tilde{Q}$ is iso. to $Q \oplus \overline{Q}$

(3) $Q$ is $F$-exact $\forall$ $F$ additive functor (covariant).

(11.5) What are ~~$F$~~ $G$-exact objects for $G = \mathrm{Hom}_{\mathcal{A}}(-,X) = h_X$ ?

Answer. – Projective objects are $G$-exact for any contra $G$ !!

Thm. $P \in \mathcal{A}$ is projective $\iff$ $\forall$ surjection $\tilde{P} \twoheadrightarrow P$,

we have a direct sum decomposition $\tilde{P} \simeq P \oplus \overline{P}$.

Remark. – In concrete situations, there are usually many more "$F$-exact" objects, which can be used as a replacement for inj/proj objects, if inj/proj do not exist.

(11.6) Big picture. – How to "compute" derived functions?

- $F : \mathcal{A} \longrightarrow \mathcal{B}$ left exact covariant.

$X \in \mathcal{A}$. Step 1. – replace $X$ by its injective resolution

$$X \rightsquigarrow Q^\bullet$$

Step 2. – apply $F$ to $Q^\bullet$ (cochain)

$$0 \to F(Q^0) \to F(Q^1) \to \cdots$$ a complex in $\mathcal{B}$

Step 3. – Take cohomology

$$R^k F(X) = \frac{\operatorname{Ker}(F(Q^k) \longrightarrow F(Q^{k+1}))}{\operatorname{Im}(F(Q^{k-1}) \longrightarrow F(Q^k))}$$

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[rr, bend left, dashed] & \text{Cochain Complexes in } \mathcal{B} \arrow[r, "H^\bullet"] & \mathcal{B} \end{tikzcd}$$

cohomology comes w/ connecting homomorphisms.

$$X \in \mathcal{A}, \quad X \rightsquigarrow Q^\bullet \rightsquigarrow F(Q^\bullet) \in \text{Cochain Complexes in } \mathcal{B}$$

- $F : \mathcal{A} \longrightarrow \mathcal{B}$ left exact <u>contravariant</u>

$$\begin{tikzcd} \mathcal{A} \arrow[r, phantom] \arrow[rr, bend left, dashed] & \text{Cochain complexes in } \mathcal{B} \arrow[r, "H"] & \mathcal{B} \end{tikzcd}$$

$$X \in \mathcal{A}, \quad X \rightsquigarrow P_\bullet \rightsquigarrow F(P_\bullet) \; (= 0 \to F(P_0) \to F(P_1) \to \cdots) \in \text{Cochain complexes in } \mathcal{B}$$

(proj. res. $\cdots \to P_k \to \cdots \to P_0 \to X \to 0$ exact in $\mathcal{A}$)

- $F : \mathcal{A} \longrightarrow \mathcal{B}$ right exact covariant

$$\begin{tikzcd}
\mathcal{A} \arrow[rrr, dashed, bend left, "H.."] & & \text{chain complexes in } \mathcal{B} \arrow[r, "\text{homology}"'] & \mathcal{B} \\
X \arrow[u, phantom, "\in" rotate=90] \arrow[r, squiggly] & P_\bullet \arrow[r, squiggly] & F(P_\bullet) \arrow[u, phantom, "\in" rotate=90] &
\end{tikzcd}$$

- $F : \mathcal{A} \longrightarrow \mathcal{B}$ right exact contravariant

$$\begin{tikzcd}
\mathcal{A} \arrow[rrr, dashed, bend left] & & \text{chain complexes in } \mathcal{B} \arrow[r, "\text{homology}"'] & \mathcal{B} \\
X \arrow[u, phantom, "\in" rotate=90] \arrow[r, squiggly] & Q^\bullet \arrow[r, squiggly] & F(Q^\bullet) \arrow[u, phantom, "\in" rotate=90] &
\end{tikzcd}$$

$$\boxed{L_k F(X) = \frac{Ker(F(P_k) \to F(P_{k-1}))}{Im(F(P_{k+1}) \to F(P_k))}}$$

(11.7) To do list. –

1. Understand how connecting homomorphisms are defined.

for $\quad H^\bullet_{(\text{or} \bullet)} :$ Complexes over $\mathcal{B} \longrightarrow \mathcal{B}$

2. Make sure we have enough injective / projective objects.

3. Make sure the dotted lines from (11.6) above do not depend on the choice of $P_\bullet$ or $Q^\bullet$.