Lecture 12

(12.0) Recall that for $X \in \mathcal{A}$ (where $\mathcal{A}$ is an abelian category) We say X is injective (resp. projective) if $\mathrm{Hom}_{\mathcal{A}}(-, X)$ (resp. $\mathrm{Hom}_{\mathcal{A}}(X, -)$) is an exact functor from $\mathcal{A}$ to $\mathcal{Ab}$ (category of abelian groups).

(12.1) Theorem. - For $Q \in \mathcal{A}$, the following conditions are equivalent.

(1) Q is injective.

(2) $\forall$ injective morphism $0 \to A \xrightarrow{f} B$ and any morphism $g : A \to Q$, $\exists\ \tilde{g} : B \to Q$ s.t. $g = \tilde{g} \circ f$ (i.e.
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "\tilde{g}"] \\ & Q & \end{tikzcd}$$
)

(3) $\forall$ injective morphism $0 \to Q \xrightarrow{i} \tilde{Q}$, $\exists\ \overline{Q} \in \mathcal{A}$ s.t.
$$\begin{tikzcd} Q \oplus \overline{Q} \arrow[rr, "\sim"] & & \tilde{Q} \\ & Q \arrow[ul, "i_1"] \arrow[ur, "i"'] & \end{tikzcd}$$

Pf. - (1) <=> (2) was already proved. The argument goes as follows:

(12.2) Theorem. – For $P \in \mathcal{A}$, TFAE: (1) P is projective

(2) $\forall$ surjective morphism $B \xrightarrow{f} C \to 0$ and $P \xrightarrow{g} C$, $\exists\, P \xrightarrow{\tilde{g}} B$ s.t. $f\tilde{g} = g$ (i.e.,

$$\begin{tikzcd} B \arrow[r] & C \arrow[r] & 0 \\ & P \arrow[ul, dashed] \arrow[u] & \end{tikzcd}$$

).

(3) $\forall$ surjective morphism $\tilde{P} \twoheadrightarrow P$, we have

$$\begin{tikzcd} \tilde{P} \arrow[r, phantom, "\cong"] \arrow[dr] & P \oplus \bar{P} \arrow[d, "pr_1"] \\ & P \end{tikzcd}$$

We will prove Thm (12.1) in this lecture (§12.5 below) and leave the (completely analoguous) proof of Thm 12.2 for the reader. We need two lemmas first.

(12.3) Lemma. Let $X \in \mathcal{A}$ and assume we have $p \in \mathrm{End}_{\mathcal{A}}(X)$ s.t. $p \circ p = p$. Then $X \simeq X_1 \oplus X_2$ where $X_1 = \mathrm{Ker}(p)$ and $X_2 = \mathrm{Im}(p)$. [Pictorially,

$$\begin{tikzcd} X \arrow[r, phantom, "\simeq"] \arrow[d, "p"] & X_1 \oplus X_2 \arrow[dr] & \\ X \arrow[r, phantom, "\simeq"] & X_1 \oplus X_2 & X_2 \arrow[l] \end{tikzcd}$$

].

Proof. – Recall that for any $Y, Y_1, Y_2 \in \mathcal{A}$, $Y \simeq Y_1 \oplus Y_2$ if, and only if we have morphisms f_1, f_2, f^1, f^2:

$$\begin{tikzcd} Y_1 \arrow[dr, "f_1"] & & Y_1 \\ & Y \arrow[ur, "f^1"] \arrow[dr, "f^2"'] & \\ Y_2 \arrow[ur, "f_2"'] & & Y_2 \end{tikzcd}$$

s.t. $f^i f_j = \delta^i_j : Y_j \to Y_i$, where $\delta^i_j = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{Id} & \text{if } i = j \end{cases}$

$$f^1 f_1 + f^2 f_2 = \mathrm{Id}_Y.$$

Now let $X_1 = \mathrm{Ker}(p) \xhookrightarrow{f_1} X$. Let $q = \mathrm{Id}_X - p \in \mathrm{End}_{\mathcal{A}}(X)$.

We have $p \circ q = q \circ p = 0$ and $q \circ q = q$.

Also, by defn. of kernel, $q \circ f_1 = f_1$.

We claim that $X_1 = \text{Image of } (q : X \to X)$, and hence we will get $X \xrightarrow{f'} X_1$ induced from q (the inclusion of the image in the target of q $(=X)$ being f_1).

Proof of the claim. As image $=$ kernel of cokernel, we need to consider $X \xrightarrow{q} X \xrightarrow{\pi} \mathrm{Coker}(q)$ and show that

$\forall\ \alpha : Z \to X$ s.t. $\pi \circ \alpha = 0$, $\exists\ \bar{\alpha} : Z \to X_1$ s.t. $f_1 \bar{\alpha} = \alpha$

By defn. of cokernel, $\pi \circ \alpha = 0 \Rightarrow \exists\ \alpha' : Z \to X$ s.t.

$$\alpha = q\alpha' = \alpha' - p\alpha'$$

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Thus $p \circ \alpha = p\alpha' - p \circ p \circ \alpha' = p\alpha' - p\alpha' = 0$

$\Rightarrow \exists\ \bar{\alpha} : Z \to X_1$ s.t. $\alpha = f_1 \bar{\alpha}$ as required.

by $X_1 = \mathrm{Ker}(p)$

$\square$ (of the pf. of the claim).

Reversing the roles of p & q above, we similarly get

$$X_2 = \mathrm{Ker}(q) \xrightarrow{f_2} X \xrightarrow{f^2} X_2 = \mathrm{Im}(p)$$

induced from p.

Easy exercise.— $\quad f^i f_j = \delta^i_j$ and $f_1 f^1 + f_2 f^2 = \mathrm{Id}_X$

(because $p + q = \mathrm{Id}_X$). $\square$

(12.4) Lemma.— Let there be given two morphisms
$$\begin{array}{ccc} A & \xrightarrow{b_1} & B_1 \\ {\scriptstyle b_2}\downarrow & & \\ B_2 & & \end{array}$$
in $\mathcal{A}$. Then: (1) $\exists\ C \in \mathcal{A}$ together with $B_\ell \xrightarrow{c_\ell} C$ $(\ell = 1,2)$ s.t. $c_1 b_1 = c_2 b_2$. Moreover, $\forall\ B_\ell \xrightarrow{d_\ell} D$ s.t. $d_1 b_1 = d_2 b_2$, $\exists!\ f : C \to D$ s.t. $f c_\ell = d_\ell$ $(\ell = 1,2)$.

(2) Assume b_1 or b_2 is injective. Then $\forall\ E \xrightarrow{e_\ell} B_\ell$ $(\ell = 1,2)$ s.t. $c_1 e_1 = c_2 e_2$, $\exists!\ e : E \to A$ s.t. $b_\ell e = e_\ell$ $(\ell = 1,2)$.

Pf. (1) Let $C = \mathrm{Coker}\left(A \xrightarrow{\begin{bmatrix} b_1 \\ -b_2 \end{bmatrix}} B_1 \oplus B_2\right)$

$$A \xrightarrow{\begin{bmatrix} b_1 \\ -b_2 \end{bmatrix}} B_1 \oplus B_2 \xrightarrow{\pi} C \to 0.$$

$c_\ell : B_\ell \to C$ are defined via
$$\begin{tikzcd} B_\ell \arrow[r, "i_\ell"] \arrow[rr, bend right, "c_\ell = \pi \circ i_\ell"'] & B_1 \oplus B_2 \arrow[r, "\pi"] & C \end{tikzcd}$$

As $\begin{bmatrix} b_1 \\ -b_2 \end{bmatrix} : A \to B_1 \oplus B_2$ is defined via universal property of finite direct sum/product, we have

$$\pi \circ \begin{bmatrix} b_1 \\ -b_2 \end{bmatrix} = 0 \implies c_1 b_1 = c_2 b_2 \quad \text{as required}$$

Now if

$$\begin{tikzcd} B_1 \arrow[dr, "d_1"] & \\ & D \\ B_2 \arrow[ur, "d_2"'] & \end{tikzcd}$$

are given so that $d_1 b_1 = d_2 b_2$ we get

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "0", "="'] & B_1 \oplus B_2 \arrow[r] \arrow[d] & C \arrow[r] \arrow[dl, dashed, "\exists! \text{ by defn. of cokernel}"] & 0 \\ & D & & \end{tikzcd}$$

(2) Assuming b_1 (or b_2) is injective, we claim that $A \xrightarrow{\begin{bmatrix} b_1 \\ -b_2 \end{bmatrix}} B_1 \oplus B_2$ is also injective (because if $Z \xrightarrow{\alpha} A \to B_1 \oplus B_2$ composition is zero then so is $Z \to A \to B_1 \oplus B_2 \to B_1$. But

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "b_1"', "="] & B_1 \oplus B_2 \arrow[d] \\ & B_1 \end{tikzcd}$$

$\implies b_1 \alpha = 0 \implies \alpha = 0$).

Thus we get that $0 \to A \to B_1 \oplus B_2 \to C \to 0$ is exact

$\implies \forall\ E \xrightarrow{\begin{bmatrix} e_1 \\ -e_2 \end{bmatrix}} B_1 \oplus B_2$ s.t. the composition $E \to C$ is zero,

$\exists!\ E \xrightarrow[e]{} A$ s.t. $b_\ell e = e_\ell$ as required. $\square$

(12.5) Proof of Theorem 12.1. — The equivalence of (1) & (2) was already proved, as follows:

(1) : Q is injective $\iff \operatorname{Hom}_{\mathcal{A}}(-, Q)$ is exact

$\iff$ $\mathrm{Hom}_{\mathcal{A}}(-, Q)$ is right exact (it is always left exact)

$\iff$ $\forall\ 0 \to A \xrightarrow{f} B$, $\quad \mathrm{Hom}(B, Q) \longrightarrow \mathrm{Hom}(A, Q)$ is surjective

which is precisely (2).

(2) $\Rightarrow$ (3). Let $Q \hookrightarrow \tilde{Q}$ be an injective morphism. We get

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, "i"] \arrow[d, equal] & \tilde{Q} \arrow[dl, dashed, "\pi"] \\ & Q & \end{tikzcd}$$

s.t. $\pi \circ i = \mathrm{Id}_Q$. Let

$p = i \circ \pi \in \mathrm{End}_{\mathcal{A}}(\tilde{Q})$. Use Lemma (12.3) to get (3).

$(p \circ p = i \circ \underbrace{\pi \circ i}_{\mathrm{Id}_Q} \circ \pi = i \circ \pi = p)$.

(3) $\Rightarrow$ (2). Given

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \\ & Q & \end{tikzcd}$$

, use Lemma 12.4 to (ii) as f is injective

get

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "q"] \\ Q \arrow[r, "h"] & \tilde{Q} \end{tikzcd}$$

. Claim. h is injective.

Pf.— $h \circ \alpha = 0$ $(\alpha : E \to Q)$ $\Rightarrow$

$$\begin{tikzcd} E \arrow[dr, dashed, "e"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\alpha"] & & \\ & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "q"] \\ & Q \arrow[r, "h"] & \tilde{Q} \end{tikzcd}$$

so $\exists\ e : E \to A$ s.t. $fe = 0 \Rightarrow e = 0$ (as f is injective)

$\Rightarrow \alpha = ge = 0$ □

Use (3), to conclude $\tilde{Q} = Q \oplus \overline{Q}$ and set $\tilde{g} = pr_1 \circ q : B \to Q$.

□

Fixing the proof of the claim on page 3.

We want to prove that $\mathrm{Ker}(p) = X_1$ is iso. to $\mathrm{Im}(q)$.

As image $= \mathrm{Ker}\left(X \xrightarrow{\pi} \mathrm{Coker}(q)\right)$ it is enough to show

that $X \xrightarrow{p} \mathrm{Im}(p)$ is the cokernel of q, i.e.

$(\forall \alpha : X \to Z)$

$pq = 0$ ✓ and $\alpha q = 0 \Rightarrow \alpha = \bar{\alpha} p$.

$$\alpha q = 0 \Rightarrow \alpha = \alpha - \alpha q = \alpha(1-q) = \alpha p$$

as required. $\square$