Lecture 13 - Tensor product

(13.0) Let R be a commutative ring with $1 \neq 0$. Let $\mathcal{A} = R\text{-mod}$ be the abelian category of R modules.

For $M, N \in \mathcal{A}$, define

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathcal{A} \\ P &\longmapsto \text{bilinear maps } \{M \times N \longrightarrow P\} \end{aligned}$$

Recall: $f : M \times N \longrightarrow P$ is an R-bilinear map if

$f(-, n) : M \longrightarrow P$ is R-linear $\forall\, n \in N$

$f(m, -) : N \longrightarrow P$ " " " $\forall\, m \in M$

Z Bilinear maps $M \times N \to P \neq \mathrm{Hom}_{\mathcal{A}}(M \times N, P)$

but $= \mathrm{Hom}_{\mathcal{A}}(N, \underbrace{\mathrm{Hom}_{\mathcal{A}}(M, P)}))$

again an R-module which is what is special about this category!

$M \otimes_R N$ is defined to be the object representing this covariant functor. That is,

(i) There is a bilinear map $M \times N \xrightarrow{i} M \underset{R}{\otimes} N$.

$m \otimes n := i(m,n)$

(ii) $\forall$ bilinear map $M \times N \xrightarrow{f} P$, $\exists!$ R-linear map $M \underset{R}{\otimes} N \xrightarrow{\tilde{f}} P$ s.t. $\tilde{f} i = f$

i.e.

$$\begin{tikzcd} M \times N \arrow[r, "i"] \arrow[d, "f"'] & M \underset{R}{\otimes} N \arrow[dl, dashed, "\tilde{f}"] \\ P & \end{tikzcd}$$

[see Lecture 29 of Algebra I (https://people.math.osu.edu/gautam.42/previous_teaching.html)

for : M : right S-module, N : left S-module $\rightsquigarrow$ $M \underset{S}{\otimes} N$, abelian group.

Most generally :

M (S_1, S)-bimod, N (S, S_2)-bimod $\rightsquigarrow$ $M \underset{S}{\otimes} N$: $(S_1 - S_2)$ bimod

— in non-commutative setting].

(13.1) Functoriality.— Given $f : M \longrightarrow M'$, $g : N \longrightarrow N'$

we get a bilinear map

$$\begin{tikzcd} M \times N \arrow[r, "(f,g)"] \arrow[dr] & M' \times N' \arrow[d, "i'"] \\ & M' \otimes N' \end{tikzcd}$$

Universal property of $M \otimes N$ gives an R-linear map, denoted by $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$

Thus we arrive at a (covariant) bifunctor

$$- \otimes - : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}.$$

Or, $\forall\, M \in \mathcal{A}$, $\quad M \otimes - : \mathcal{A} \longrightarrow \mathcal{A}$.

(13.2) Easy properties. – left as an exercise. [follow from universal prop.]

(i) $M \otimes N \simeq N \otimes M$

(ii) $M \otimes (N \otimes P) = (M \otimes N) \otimes P$

(iii) $M \otimes -$ is an additive functor. Hence,

$$M \otimes \Big(\bigoplus_{i \in I} N_i\Big) \simeq \bigoplus_{i \in I} M \otimes N_i$$

(13.3) Existence of $M \otimes_R N$. –

Given $M, N \in \mathcal{A} = R\text{-mod}$, define the free abelian group $\mathbb{Z}^{(M \times N)}$ as the direct sum $\bigoplus_{i \in M \times N} X_i$, where $X_i = \mathbb{Z}$ $\forall i$

Notation.— $\forall (m,n) \in M \times N$; $T(m,n) \in \mathbb{Z}^{(M\times N)}$ is the element with coordinates 0 everywhere, except at $(m,n)^{th}$ component; where its entry is 1.

Let $K \subset \mathbb{Z}^{(M\times N)}$ be subgroup generated by

$$\cancel{T(r_1m_1+r_2m_2,n) - T(r_1m_1,n) - T(r_2m_2,n)}$$

$$T(m_1+m_2,n) - T(m_1,n) - T(m_2,n)$$

$$T(m,n_1+n_2) - T(m,n_1) - T(m,n_2)$$

$$T(rm,n) - T(m,rn)$$

$\forall$ $r \in R$, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$

$$\boxed{M \otimes_R N := \mathbb{Z}^{(M\times N)} / K}$$

Lemma. $\mathbb{Z}^{(M\times N)}/K$, together with

$$M \times N \xrightarrow{\ i\ } \mathbb{Z}^{(M\times N)}/K$$

$$(m,n) \longmapsto T(m,n) \bmod K$$

$$(= m \otimes n)$$

satisfies the U.P. of the tensor product.

Proof. – i is bilinear (over R) :– here R-module structure on $\mathbb{Z}^{(M\times N)}/K$ is given by $r.T(m,n) = T(rm,n)$

$(= T(m,r.n) \bmod K)$

Bilinearity of i follows from the defn. of K.

Given $f: M\times N \longrightarrow P$; an R-bilinear map; we get

$$\mathbb{Z}^{(M\times N)} \xrightarrow{F} P \quad \Big(F\Big(\sum_{\text{finite}} a_{m,n} T(m,n)\Big) = \sum_{\text{finite}} a_{m,n} f(m,n)$$

$(a_{m,n} \in \mathbb{Z})$

f is bilinear $\Rightarrow$ F factors through K and we get an R-linear map $\tilde{f} : \mathbb{Z}^{(M\times N)}/K \longrightarrow P$ with the required property $\tilde{f}\circ i = f$ $\square$

(13.4) In practice – a typical element of $M\otimes N$ looks like a <u>finite sum</u> $\sum_{j=1}^{N} m_j \otimes n_j$.

Rules of manipulation :

$$(m_1+m_2)\otimes n = m_1\otimes n + m_2\otimes n$$

$$m\otimes(n_1+n_2) = m\otimes n_1 + m\otimes n_2$$

$$(rm)\otimes n = \underbrace{r(m\otimes n)}_{} = m\otimes(rn)$$

defn of this term

(13.5) Some examples from Algebra I

$$\mathbb{Z}/_{m\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}/_{n\mathbb{Z}} = \mathbb{Z}/_{\gcd(m,n)\mathbb{Z}}$$

$R \underset{R}{\otimes} N = N$. More generally $R^{(I)} \underset{R}{\otimes} N = N^{(I)}$.

($\bigoplus_{i \in I} X_i$ where $X_i = R$ as an R-mod)

(13.6) Theorem $M \otimes - : \mathcal{A} \longrightarrow \mathcal{A}$ is right exact.

but not left exact, e.g. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ is inj. but

$\mathbb{Z}/_{2\mathbb{Z}} \otimes - \; : \; \mathbb{Z}/_{2\mathbb{Z}} \xrightarrow{0} \mathbb{Z}/_{2\mathbb{Z}}$ is not!

Pf. Let $0 \to N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0$ be a short exact seq. of R-modules

To show: (i) $M \otimes N_2 \xrightarrow{1 \otimes g} M \otimes N_3$ is surjective

(because, each $m \otimes n_3 = (1 \otimes g)(m \otimes n_2)$ where $g(n_2) = n_3$ exist by surjectivity of g. As $M \otimes N_3$ is generated by $\left\{ m \otimes n_3 : \begin{array}{l} m \in M \\ n_3 \in N_3 \end{array} \right\}$ as an abelian gp., we get surjectivity of $1 \otimes g$).

(ii) $\mathrm{Im}(1\otimes f) \subset \mathrm{Ker}(1\otimes g)$ (because $(1\otimes g)\circ(1\otimes f)$ $= 1\otimes(g\circ f) = 1\otimes(\text{zero morphism}) = 0$, by additivity of $M\otimes -$).

(iii) Now consider $\dfrac{M\otimes N_2}{\mathrm{Im}(1\otimes f)} \xrightarrow{\tilde{g}} M\otimes N_3$. We'll

induced from $1\otimes g$

construct its inverse, showing that $\tilde{g}$ is an iso, hence $\mathrm{Ker}(1\otimes g) = \mathrm{Im}(1\otimes f)$. So, let

$$M \times N_3 \xrightarrow{h} M\otimes N_2 / \mathrm{Im}(1\otimes f)$$

$$(m, n_3) \longmapsto (m\otimes n_2) \bmod \mathrm{Im}(1\otimes f)$$ for some n_2 s.t. $g(n_2) = n_3$. If we pick a different n_2', we will get $n_2 - n_2' \in \mathrm{Ker}(g) = \mathrm{Im}(f) \Rightarrow m\otimes n_2 \equiv m\otimes n_2' \bmod \mathrm{Im}(1\otimes f)$.

Check. – h is R-bilinear

Hence we get a well-defined R-linear map $M\otimes N_3 \xrightarrow{\tilde{h}} \dfrac{M\otimes N_2}{\mathrm{Im}(1\otimes f)}$

It is (very) easy to show that $\tilde{h}\,\tilde{g} = \mathrm{Id}$ & $\tilde{g}\circ\tilde{h} = \mathrm{Id}$.

$\square$