Lecture 14

(14.0) Let R be a commutative ring with $1 \neq 0$. Let $\mathcal{A} = R\text{-mod}$ be the (abelian) category of modules over R. Our first goal is to understand cohomology functors on the category of (cochain) complexes of R-modules.

As mentioned earlier, homological algebra was formalized in 1940's (Cartan, Eilenberg, MacLane, Grothendieck...). However, in different guises, methods of homological algebra were present in the earlier works such as.– (i) Hilbert's syzygy theorems (1890); (ii) Schur's projective repns. (1904); (iii) Group extensions à la Schrier (1926); (iv) Cross product algebras of Noether (1929).

(14.1) Category of (cochain) complexes. – Let $\mathbb{K}^\bullet(\mathcal{A})$ denote the category whose objects are $(X^\bullet, d^\bullet)$ where

- $X^\bullet = \{X^n \in \mathcal{A}\}_{n\in\mathbb{Z}}$
- $d^n : X^n \longrightarrow X^{n+1}$ $\forall\, n \in \mathbb{Z}$ are morphisms of $\mathcal{A}$

(i.e., R-linear homomorphisms) such that

$$d^{n+1} \circ d^n = 0 \quad \forall\, n \in \mathbb{Z}. \qquad \left[\text{called "differential".}\right]$$

Morphisms $(X^\bullet, d_X^\bullet) \longrightarrow (Y^\bullet, d_Y^\bullet)$ are collections

$\alpha^\bullet = \{\alpha^n : X^n \to Y^n\}_{n\in\mathbb{Z}}$ such that

$$\begin{tikzcd} X^n \arrow[r, "\alpha^n"] \arrow[d, "d_X^n"'] & Y^n \arrow[d, "d_Y^n"] \\ X^{n+1} \arrow[r, "\alpha^{n+1}"'] & Y^{n+1} \end{tikzcd}$$

commutes. $(\forall n \in \mathbb{Z})$

i.e. (a bit imprecisely)

$$\boxed{d \circ \alpha = \alpha \circ d}$$

Remarks. — 1. On terminology; a chain complex, similarly has integer subscripts (as opposed to superscripts) and d decreases the label by 1. (i.e. $\cdots X_4 \xrightarrow{d_4} X_3 \xrightarrow{d_3} X_2 \cdots$). As mentioned earlier, there is no fundamental difference between the two concepts.

2. $\mathbb{K}^\bullet(\mathcal{A})$ is automatically an abelian category because of Problem 1 of Homework 4. More explicitly,

- $\mathrm{Hom}_{\mathbb{K}^\bullet(\mathcal{A})}((X^\bullet, d^\bullet), (Y^\bullet, d^\bullet)) \subset \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^n, Y^n)$

  is an (abelian) subgroup.

- $\mathrm{Ker}(\alpha^\bullet : X^\bullet \to Y^\bullet) =: K^\bullet$ is defined by

  $$K^n = \mathrm{Ker}(\alpha^n : X^n \to Y^n) \hookrightarrow X^n$$

  d_X^n restricted to K^n lands in K^{n+1} and thus defines

  $d_K^n : K^n \to K^{n+1}$.

- Direct sums (and products) are taken componentwise as well. For example, $C^\bullet = A^\bullet \oplus B^\bullet$ is defined as:
$$C^n = A^n \oplus B^n \ ; \ d_C^n = d_A^n \oplus d_B^n \quad \forall\, n \in \mathbb{Z}.$$

(14.2) Cohomology functors. – Let $(X^\bullet, d^\bullet) \in \mathbb{K}^\bullet(\mathcal{A})$.

Elements of $\mathrm{Ker}(d^n)$ are called <u>n- cocycles</u>.
Elements of $\mathrm{Image}(d^{n-1})$ are called <u>n- coboundries</u>.

$$H^n(X^\bullet) := \mathrm{Ker}(d^n) \big/ \mathrm{Im}(d^{n-1}) \qquad n^{th} \text{ cohomology of } X^\bullet.$$

Proposition. – For each $n \in \mathbb{Z}$, $H^n : \mathbb{K}^\bullet(\mathcal{A}) \longrightarrow \mathcal{A}$ is an additive, semi-exact functor (i.e., $\forall$ short exact sequence $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ in $\mathbb{K}^\bullet(\mathcal{A})$, we get an exact seq.
$$H^n(X^\bullet) \longrightarrow H^n(Y^\bullet) \longrightarrow H^n(Z^\bullet) \ )$$

(14.3) Theorem. – For every short exact sequence
$$0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0 \quad \text{in } \mathbb{K}^\bullet(\mathcal{A}); \text{ and}$$
$\forall\, n \in \mathbb{Z}$, there is a morphism
$$\delta : H^n(Z^\bullet) \longrightarrow H^{n+1}(X^\bullet) \quad \text{such that}$$

(i)
$$\begin{tikzcd} \cdots \arrow[r] & H^n(X^\bullet) \arrow[r] & H^n(Y^\bullet) \arrow[r] & H^n(Z^\bullet) \arrow[dll, "\delta_n"] \\ & H^{n+1}(X^\bullet) \arrow[r] & H^{n+1}(Y^\bullet) \arrow[r] & H^{n+1}(Z^\bullet) \cdots \end{tikzcd}$$
is exact in $\mathcal{A}$

(ii) For every

$$\begin{tikzcd} 0 \arrow[r] & X_1^\bullet \arrow[r] \arrow[d] & Y_1^\bullet \arrow[r] \arrow[d] & Z_1^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X_2^\bullet \arrow[r] & Y_2^\bullet \arrow[r] & Z_2^\bullet \arrow[r] & 0 \end{tikzcd}$$

in $\mathbb{K}^\bullet(\mathcal{A})$

where the rows are exact and two squares commute, the following diagram is commutative $(\forall n \in \mathbb{Z})$

$$\begin{tikzcd} H^n(Z_1^\bullet) \arrow[r] \arrow[d] & H^{n+1}(X_1^\bullet) \arrow[d] \\ H^n(Z_2^\bullet) \arrow[r] & H^{n+1}(X_2^\bullet) \end{tikzcd}$$

(14.4) Proofs of Proposition (14.2) and Theorem 14.3 rely on the famous Snake Lemma.

Snake Lemma. - Consider the following commutative diagram with exact rows: (of R-modules)

$$\begin{tikzcd} M_1 \arrow[r, "u"] \arrow[d, "f"'] & M_2 \arrow[r, "v"] \arrow[d, "g"] & M_3 \arrow[d, "h"] \\ N_1 \arrow[r, "u'"'] & N_2 \arrow[r, "v'"'] & N_3 \end{tikzcd}$$

which then gives rise to a commutative diagram:

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "u_1"] \arrow[d, "i"'] & \mathrm{Ker}(g) \arrow[r, "v_1"] \arrow[d, "j"'] & \mathrm{Ker}(h) \arrow[d, "k"] \\ M_1 \arrow[r, "u"] \arrow[d, "f"'] & M_2 \arrow[r, "v"] \arrow[d, "g"] & M_3 \arrow[d, "h"] \\ N_1 \arrow[r, "u'"] \arrow[d, "p"'] & N_2 \arrow[r, "v'"] \arrow[d, "q"] & N_3 \arrow[d, "r"] \\ \mathrm{Coker}(f) \arrow[r, "u_2"'] & \mathrm{Coker}(g) \arrow[r, "v_2"'] & \mathrm{Coker}(h) \end{tikzcd}$$

(i) $v_1 \circ u_1 = 0$. If u' is injective, the following seq. is exact

$$\mathrm{Ker}(f) \xrightarrow{u_1} \mathrm{Ker}(g) \xrightarrow{v_1} \mathrm{Ker}(h)$$

(ii) $v_2 \circ u_2 = 0$. If v is surjective, (u_2, v_2) is exact.

(iii) Assume u' is injective & v is surjective. Then we get

$\delta : \mathrm{Ker}(h) \longrightarrow \mathrm{Coker}(f)$ making the following exact:

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "u_1"] & \mathrm{Ker}(g) \arrow[r, "v_1"] & \mathrm{Ker}(h) \arrow[r, "\delta"] & \mathrm{Coker}(f) \arrow[r, "u_2"] & \mathrm{Coker}(g) \arrow[d, "v_2"] \\
 & & & & \mathrm{Coker}(h)
\end{tikzcd}$$

We often see this lemma sketched pictorially as

$$\begin{tikzcd}
 & \mathrm{Ker}(f) \arrow[r] \arrow[d] & \mathrm{Ker}(g) \arrow[r] \arrow[d] & \mathrm{Ker}(h) \arrow[d] \arrow[dddll, dashed, bend left, "\delta"] & \\
 & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d] & N_3 \arrow[d] & \\
 & \mathrm{Coker}(f) \arrow[r] & \mathrm{Coker}(g) \arrow[r] & \mathrm{Coker}(h) &
\end{tikzcd}$$

Proof. (i) $k \circ (v_1 \circ u_1) = (v \circ u) \circ i = 0$ and k is injective

$\Rightarrow v_1 \circ u_1 = 0$. Hence $\mathrm{Im}(u_1) \subset \mathrm{Ker}(v_1)$

Now assume that u' is injective. To prove: $\mathrm{Ker}(v_1) \subset \mathrm{Im}(u_1)$.

Let $x \in \mathrm{Ker}(g)$ s.t. $v_1(x) = 0$. Then

- $m_2 := j(x)$ satisfies $v(m_2) = v(j(x)) = k(v_1(x)) = 0$

  $\Rightarrow \exists\, m_1 \in M_1$ s.t. $m_2 = u(m_1)$.

- $n_1 := f(m_1)$ satisfies $u'(n_1) = 0$ (because, we have

  $u'(n_1) = u'(f(m_1)) = g(u(m_1)) = g(m_2) = g(j(x)) = 0$)

Since u' is injective, we get $n_1 = f(m_1) = 0 \Rightarrow m_1 = i(y)$ for (unique) $y \in \mathrm{Ker}(f)$. Let $x' = u_1(y) \in \mathrm{Ker}(g)$. Then

$$j(x') = j(u_1(y)) = u(i(y)) = u(m_1) = m_2 = j(x).$$

As j is injective, we get $x = x' = u_1(y) \Rightarrow x \in \mathrm{Image}(u_1)$ as required.

(ii) is proved similarly and we leave the details to the reader.

(iii) Now let $x \in \mathrm{Ker}(h)$ and we will define $\delta(x) \in \mathrm{Coker}(f)$.

The definition follows the same "diagram chase":

- $m_3 := k(x) \in M_3$. As v is surjective, $\exists\, m_2 \in M_2$

[Choice!] s.t. $v(m_2) = m_3$

- set $n_2 := g(m_2) \in N_2$. This element satisfies

$$v'(n_2) = v'(g(m_2)) = h(v(m_2)) = h(m_3)$$
$$= h(k(x)) = 0.$$ Thus (by exactness of $0 \to N_1 \to N_2 \to N_3$)

$\exists!\ n_1 \in N_1$ s.t. $n_2 = u'(n_1)$.

Define $\delta(x) := p(n_1) \in \mathrm{Coker}(f)$.

Check. – $\delta(x)$ does not depend on the [Choice!] choice of $m_2 \in M_2$ made in first step.

Proof. – Let $\tilde{m}_2$ be another choice. (i.e. $v(\tilde{m}_2) = m_3$).

Following the recipe of defining δ, with tildes, we have:

- $v(m_2) = v(\tilde{m}_2) = m_3 \Rightarrow v(m_2 - \tilde{m}_2) = 0 \Rightarrow \exists\ \lambda \in M_1$ s.t. $u(\lambda) = m_2 - \tilde{m}_2$.

- $n_2 = g(m_2)$, $\exists!\ n_1 \in N_1$ s.t. $n_2 = u'(n_1)$
  $\tilde{n}_2 = g(\tilde{m}_2)$, $\exists!\ \tilde{n}_1 \in N_1$ s.t. $\tilde{n}_2 = u'(\tilde{n}_1)$

Then $u'(n_1 - \tilde{n}_1) = n_2 - \tilde{n}_2 = g(m_2 - \tilde{m}_2) = g(u(\lambda))$
$$= u'(f(\lambda))$$

u' injective $\Rightarrow n_1 - \tilde{n}_1 = f(\lambda)$

Hence $\delta(x) - \widetilde{\delta(x)} = p(n_1) - p(\tilde{n}_1) = p(f(\lambda)) = 0$.

We are now left with the verification of : (a) δ is R-linear

(b) $\delta \circ v_1 = 0$ (c) $\mathrm{Ker}(\delta) \subset \mathrm{Im}(v_1)$.

Proof of (a) : Let $x, x' \in \mathrm{Ker}(h)$ and $r, r' \in R$. To prove :

$$\delta(rx + r'x') = r\,\delta(x) + r'\delta(x').$$

Let $m_2, m_2' \in M_2$ be the choices made for the defn. of $\delta(x)$ and $\delta(x')$ resp., at the first step. Set $m_2'' = rm_2 + r'm_2' \in M_2$. Then $v(m_2'') = r\,v(m_2) + r'v(m_2') = k(rx + r'x')$, meaning m_2'' can be used as a choice at the first step to define $\delta(rx + r'x')$. Following the recipe for the defn. of δ, we easily see that $\delta(rx + r'x') = r\,\delta(x) + r'\delta(x')$.

Proof of (b) : If $x = v_1(y)$ for some $y \in \mathrm{Ker}(g)$, we can take $m_2 = j(y)$ in the 1^{st} step (because $v(m_2) = v(j(y))$ $= k(v_1(y)) = k(x)$).

But then $n_2 = g(m_2) = g(j(y)) = 0$ and hence $n_1 = 0$ and consequently $\delta(x) = p(n_1) = 0$.

Proof of (c) : Let $x \in \mathrm{Ker}(h)$ be such that $\delta(x) = 0$. Let m_2, n_2, n_1 be the elements as introduced in the defn. of $\delta(x)$. Then

$$p(n_1) = \delta(x) = 0 \implies n_1 \in \mathrm{Im}(f), \text{ i.e. } \exists\, m_1 \in M_1 \text{ s.t. } f(m_1) = n_1$$

But then $u(m_1) - m_2 \in M_2$ satisfies

$$g(u(m_1) - m_2) = g(u(m_1)) - g(m_2) = u'(f(m_1)) - n_2$$
$$= u'(n_1) - n_2 = 0$$

$\Rightarrow \exists\, y \in \operatorname{Ker}(g)$ s.t. $j(y) = -u(m_1) + m_2$.

Claim.- $v_1(y) = x$.

Pf. of the claim.- $k(v_1(y)) = v(j(y))$

$$= v(m_2) - \underbrace{v(u(m_1))}_{0} = k(x).$$ As k is injective, we get $v_1(y) = x$ as required. $\square$