

Lecture 15

①

(15.0) Recall: last time we defined the category of cochain complexes of R -modules, denoted by $K^*(A)$ where $A = R\text{-mod}$ & R is a commutative ring (with $1 \neq 0$).

Our goal is to prove Prop. 14.2 and Theorem 14.3. As mentioned last time, the proof relies on the Snake Lemma (§14.4).

Snake Lemma. Given a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 M_1 & \xrightarrow{u} & M_2 & \xrightarrow{v} & M_3 & \rightarrow & 0 \\
 f \downarrow & & \downarrow g & & \downarrow h & & \\
 0 \rightarrow N_1 & \xrightarrow{u'} & N_2 & \xrightarrow{v'} & N_3 & &
 \end{array}$$

we get $\delta: \text{Ker}(h) \rightarrow \text{Coker}(f)$

such that $\text{Ker}(f) \xrightarrow{u_1} \text{Ker}(g) \xrightarrow{v_1} \text{Ker}(h) \xrightarrow{\delta} \text{Coker}(f) \xrightarrow{u_2} \text{Coker}(g) \xrightarrow{v_2} \text{Coker}(h)$ is exact.

(15.1) Set up. (for Prop. 14.2). $n \in \mathbb{Z}$ is fixed.

$\forall C^\bullet \in K^*(A)$, let us denote by $Z^n(C^\bullet) = \text{Ker}(d^n)$

$$(\dots \rightarrow C^n \xrightarrow{d^n} C^{n+1} \rightarrow \dots)$$

and by $B^n(C^\bullet) = \text{Image}(d^{n-1})$

Last time we defined $H^n(C^\bullet) = Z^n(C^\bullet) / B^n(C^\bullet)$

Thus we defined $H^n : K(A) \rightarrow A$ on objects. (2)

Now let $\alpha : C \rightarrow D$ be a morphism of complexes.

We get (i)

$$\begin{array}{ccc} C^n & \xrightarrow{d_C^n} & C^{n+1} \\ \alpha^n \downarrow & & \downarrow \alpha^{n+1} \\ D^n & \xrightarrow{d_D^n} & D^{n+1} \end{array} \quad \forall x \in \mathbb{Z}^n(C) \text{ i.e.} \\ & & x \in C^n \text{ s.t. } d(x) = 0$$

we have $d_D^n \alpha^n(x) = \alpha^{n+1}(d_C^n(x)) = 0$

$\Rightarrow \alpha^n : \text{Ker}(d_C^n) = \mathbb{Z}^n(C) \rightarrow \mathbb{Z}^n(D) = \text{Ker}(d_D^n).$

(ii) Similarly α^n restricts to a homomorphism from $B^n(C)$ to $B^n(D)$; i.e.

$$\begin{array}{ccc} B^n(C) \subset \mathbb{Z}^n(C) & & H^n(C) \\ \downarrow & & \downarrow H^n(\alpha) \text{ [defn.]} \\ B^n(D) \subset \mathbb{Z}^n(D) & \Rightarrow & H^n(D) \end{array}$$

Exercise (easy). $H^n(\text{Id}_C) = \text{Id}_{H^n(C)}$; $H^n(\alpha + \alpha') = H^n(\alpha) + H^n(\alpha')$

$$H^n(\beta \circ \alpha) = H^n(\beta) \circ H^n(\alpha)$$

The semi-exactness of H^n (last claim of Prop. 14.2) is contained in the assertion of Theorem 14.3 (i) and will be proved there.

(15.2) Now assume we have a short exact sequence

of complexes $0 \rightarrow C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \rightarrow 0$

Meaning there is a giant grid (only many exact rows; 3 columns)

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \rightarrow & C_1 & \xrightarrow{\alpha^n} & C_2 & \xrightarrow{\beta^n} & C_3 \rightarrow 0 \\
 & & \downarrow d_1^n & & \downarrow d_2^n & & \downarrow d_3^n \\
 0 & \rightarrow & C_1^{n+1} & \xrightarrow{\alpha^{n+1}} & C_2^{n+1} & \xrightarrow{\beta^{n+1}} & C_3^{n+1} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

all squares commute!

(i) Using Snake Lemma (with $M_l = C_l^{n+1}$ & $N_l = C_l^{n+2}$)
($l = 1, 2, 3$)

and using the fact that α^{n+1} & α^n are injective; we get an exact sequence

$$0 \rightarrow Z^{n+1}(C_1) \rightarrow Z^{n+1}(C_2) \rightarrow Z^{n+1}(C_3)$$

(ii) Similarly we get an exact sequence

$$\begin{array}{ccccc}
 C_1^n / B^n(C_1) & \rightarrow & C_2^n / B^n(C_2) & \rightarrow & C_3^n / B^n(C_3) \rightarrow 0 \\
 \parallel & & \parallel & & \parallel \\
 \text{Coker}(d_1^{n-1}) & & \text{Coker}(d_2^{n-1}) & & \text{Coker}(d_3^{n-1})
 \end{array}$$

(15.3) Naturality of connecting homomorphism (part (ii) of Theorem 14.3).

Now assume that we are given two short exact sequences and a commutative diagram between them.

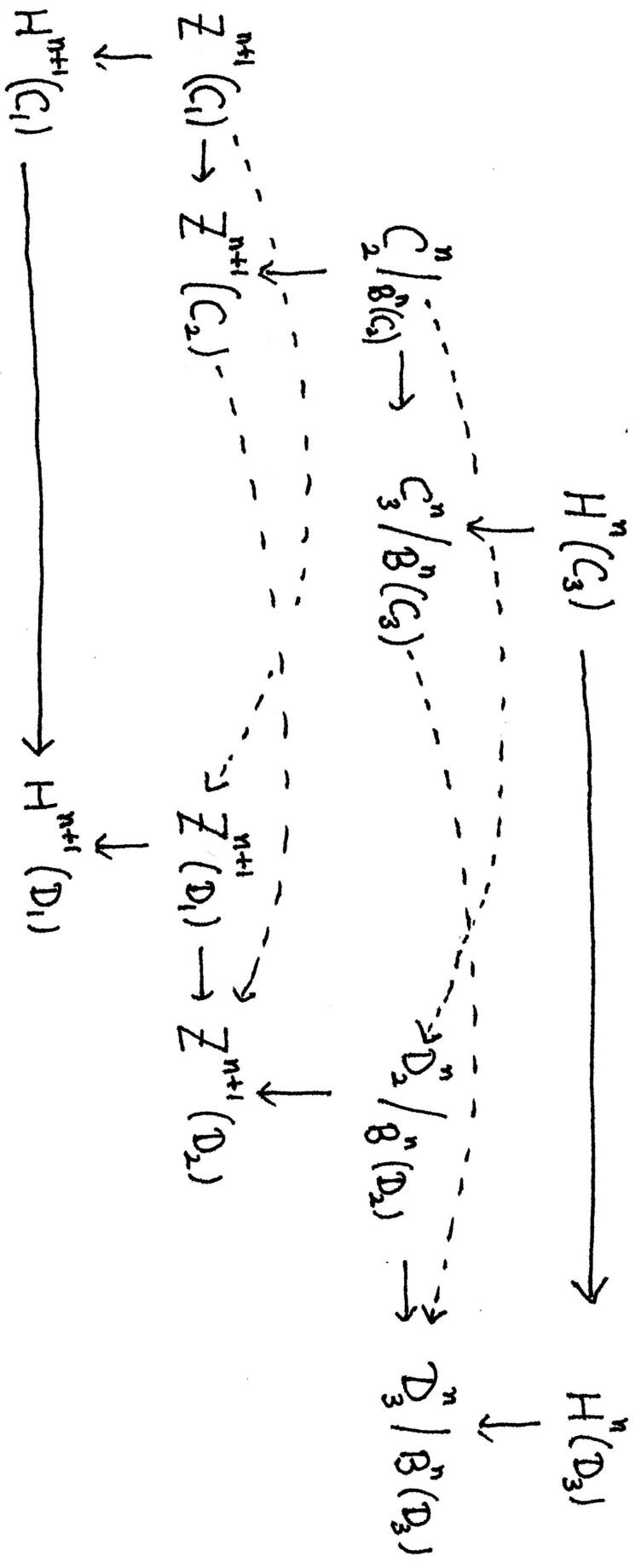
$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 \longrightarrow 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h \\
 0 & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow & D_3 \longrightarrow 0
 \end{array}$$

and we want to prove the commutativity of

$$\begin{array}{ccc}
 H^n(C_3) & \xrightarrow{\delta_C^n} & H^{n+1}(C_1) \\
 \downarrow H^n(h) & & \downarrow H^{n+1}(f) \\
 H^n(D_3) & \xrightarrow{\delta_D^n} & H^{n+1}(D_1)
 \end{array}$$

We will use the diagram of page 4 - defn. of δ via Snake Lemma. The commutativity of $H^{n+1}(f) \delta_C^n \stackrel{?}{=} \delta_D^n H^n(h)$ follows from that of each smaller piece: (see big cartoon on next page)

$$\begin{array}{ccc}
 H^n(C_3) \longrightarrow C_2^n/B^n(C_2) \longrightarrow C_3^n/B^n(C_3) & & \\
 \downarrow & \downarrow & \downarrow \\
 H^n(D_3) \longrightarrow D_2^n/B^n(D_2) \longrightarrow D_3^n/B^n(D_3) & & \\
 \\
 C_2^n/B^n(C_2) \longrightarrow Z^{n+1}(C_2) & & H^{n+1}(C_1) \longleftarrow Z^{n+1}(C_1) \longrightarrow Z^{n+1}(C_2) \\
 \downarrow & \downarrow & \downarrow \\
 D_2^n/B^n(D_2) \longrightarrow Z^{n+1}(D_2) & & H^{n+1}(D_1) \longleftarrow Z^{n+1}(D_1) \longrightarrow Z^{n+1}(D_2)
 \end{array}$$

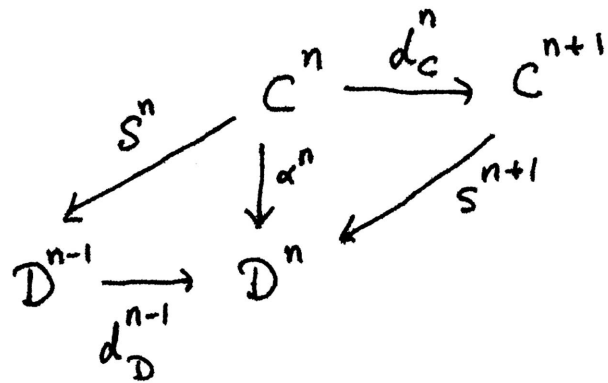


Dessin d'enfants

(15.4) Homotopy. - Let C^\bullet and D^\bullet be two cochain complexes and let $\alpha^\bullet: C^\bullet \rightarrow D^\bullet$ be a morphism. We

say α is null-homotopic ($\alpha \sim 0$) if $\exists s^n: C^n \rightarrow D^{n-1}$ ($\forall n \in \mathbb{Z}$)

s.t. $\alpha = ds + sd$ (w/ appropriate superscripts)



As usual, if $\beta, \gamma \in \text{Hom}_{K(A)}(C^\bullet, D^\bullet)$, we say $\beta \sim \gamma$

(β is homotopic to γ) if $\beta - \gamma$ is null-homotopic (i.e. we can find $s^n: C^n \rightarrow D^{n-1}$ s.t. $\beta - \gamma = ds + sd$) for every $n \in \mathbb{Z}$

Proposition. - (i) $\text{Hom}_{K(A)}^{\text{null}}(C^\bullet, D^\bullet)$ is an abelian subgroup [null-homotopic maps] of $\text{Hom}_{K(A)}(C^\bullet, D^\bullet)$

(ii) $\alpha^\bullet \sim 0 \Rightarrow \beta \alpha \sim 0$ and $\alpha \gamma \sim 0$
 $(\alpha^\bullet: C^\bullet \rightarrow D^\bullet) \quad \forall \beta: D^\bullet \rightarrow E^\bullet \quad \forall \gamma: A^\bullet \rightarrow C^\bullet$

[analogue of "ideals in rings" - good for modding out!]

(iii) $\alpha \sim 0 \Rightarrow H^n(\alpha) = 0 \quad \forall n \in \mathbb{Z}.$

Proof. - (i) $\alpha \sim 0 \Rightarrow \exists s^n: C^n \rightarrow D^{n-1}$ such that $\alpha = ds + sd.$

Then $(-\alpha) = d(-s) + (-s)d \Rightarrow -\alpha \sim 0.$

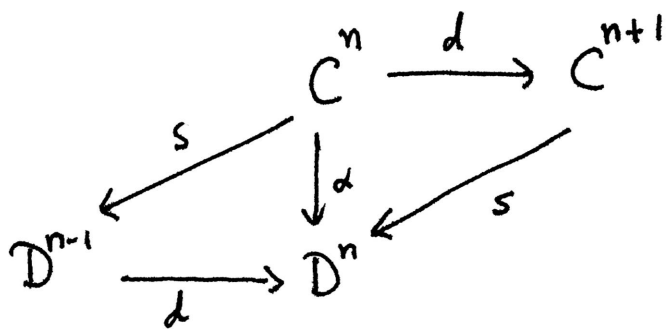
$\alpha + \alpha' = d(s+s') + (s+s')d \Rightarrow \alpha + \alpha' \sim 0.$

(ii) Assume $C \xrightarrow{\alpha} D \xrightarrow{\beta} E$ & $\alpha \sim 0$ (i.e. $\alpha = ds + sd$)

Then $\beta\alpha = \beta(ds + sd) = d(\beta s) + (\beta s)d$ ($\beta d = d\beta$ by defn.)
 $\Rightarrow \beta\alpha \sim 0.$

Same argument for $\alpha\gamma \sim 0.$

(iii) Again if $\alpha: C \rightarrow D$ and $\alpha \sim 0$ (i.e. $\alpha = ds + sd$ for some s)



$\forall x \in \text{Ker}(d_C^n)$

$\alpha(x) = ds(x) + sd(x) \rightarrow 0$
 $\in \text{Image of } (d_D^{n-1})$

$\Rightarrow H^n(\alpha) = 0.$

□