Lecture 15

(15.0) Recall: last time we defined the category of cochain complexes of R-modules., denoted by $\mathbb{K}^\bullet(\mathcal{A})$ where $\mathcal{A} = R\text{-mod}$ & R is a commutative ring (with $1 \neq 0$).

Our goal is to prove Prop. 14.2 and Theorem 14.3. As mentioned last time, the proof relies on the Snake Lemma (§14.4).

Snake Lemma. Given a commutative diagram with exact rows

$$\begin{tikzcd} & M_1 \arrow[r, "u"] \arrow[d, "f"'] & M_2 \arrow[r, "v"] \arrow[d, "g"'] & M_3 \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & N_1 \arrow[r, "u'"'] & N_2 \arrow[r, "v'"'] & N_3 & \end{tikzcd}$$

we get $\delta : \mathrm{Ker}(h) \to \mathrm{Coker}(f)$

such that

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "u_1"] & \mathrm{Ker}(g) \arrow[r, "v_1"] & \mathrm{Ker}(h) \arrow[dll, "\delta", rounded corners] \\ \mathrm{Coker}(f) \arrow[r, "u_2"'] & \mathrm{Coker}(g) \arrow[r, "v_2"'] & \mathrm{Coker}(h) \end{tikzcd}$$

is exact.

(15.1) Set up. (for Prop. 14.2). $n \in \mathbb{Z}$ is fixed.

$\forall\, C^\bullet \in \mathbb{K}^\bullet(\mathcal{A})$, let us denote by $Z^n_*(C^\bullet) = \mathrm{Ker}(d^n)$

$(\cdots \to C^n \xrightarrow{d^n} C^{n+1} \to \cdots)$

and by $B^n_*(C^\bullet) = \mathrm{Image}(d^{n-1})$

Last time we defined $H^n(C^\bullet) = Z^n(C^\bullet) / B^n(C^\bullet)$

Thus we defined $H^n : \mathbb{K}^\bullet(\mathcal{A}) \longrightarrow \mathcal{A}$ on objects.

Now let $\alpha^\bullet : C^\bullet \longrightarrow D^\bullet$ be a morphism of complexes.

We get (i)

$$\begin{tikzcd} C^n \arrow[r, "d_C^n"] \arrow[d, "\alpha^n"'] & C^{n+1} \arrow[d, "\alpha^{n+1}"] \\ D^n \arrow[r, "d_D^n"'] & D^{n+1} \end{tikzcd}$$

$\forall\, x \in Z^n(C^\bullet)$ i.e. $x \in C^n$ s.t. $d(x) = 0$

we have $d_D^n \alpha^n(x) = \alpha^{n+1}(d_C^n(x)) = 0$

$$\Rightarrow \quad \alpha^n : \mathrm{Ker}(d_C^n) = Z^n(C) \longrightarrow Z^n(D) = \mathrm{Ker}(d_D^n).$$

(ii) Similarly α^n restricts to a homomorphism from $B^n(C)$ to $B^n(D)$; i.e.

$$\begin{tikzcd} B^n(C) \arrow[r, phantom, "\subset"] \arrow[d] & Z^n(C) \arrow[d] \\ B^n(D) \arrow[r, phantom, "\subset"] & Z^n(D) \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} H^n(C) \arrow[d, "H^n(\alpha)\ \text{[defn.]}"] \\ H^n(D) \end{tikzcd}$$

Exercise (easy). $H^n(\mathrm{Id}_C) = \mathrm{Id}_{H^n(C)}$; $H^n(\alpha + \alpha') = H^n(\alpha) + H^n(\alpha')$

$$H^n(\beta \circ \alpha) = H^n(\beta) \circ H^n(\alpha)$$

The semi-exactness of H^n (last claim of Prop. 14.2) is contained in the assertion of Theorem 14.3 (i) and will be proved there.

(15.2) Now assume we have a short exact sequence of complexes $0 \longrightarrow C_1^\bullet \xrightarrow{\alpha} C_2^\bullet \xrightarrow{\beta} C_3^\bullet \longrightarrow 0$

Meaning there is a giant grid (∞ly many exact rows; 3 columns)

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1^n & \xrightarrow{\alpha^n} & C_2^n & \xrightarrow{\beta^n} & C_3^n & \longrightarrow & 0 \\
 & & {\scriptstyle d_1^n}\downarrow & & \downarrow{\scriptstyle d_2^n} & & \downarrow{\scriptstyle d_3^n} & & \\
0 & \longrightarrow & C_1^{n+1} & \xrightarrow[\alpha^{n+1}]{} & C_2^{n+1} & \xrightarrow[\beta^{n+1}]{} & C_3^{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

all squares commute!

(i) Using Snake Lemma (with $M_\ell = C_\ell^{n+1}$ & $N_\ell = C_\ell^{n+2}$)
$(\ell = 1, 2, 3)$

and using the fact that α^{n+1} & α^n are injective; we get an exact sequence

$$0 \longrightarrow Z^{n+1}(C_1) \longrightarrow Z^{n+1}(C_2) \longrightarrow Z^{n+1}(C_3)$$

(ii) Similarly we get an exact sequence

$$\begin{array}{ccccccc}
C_1^n / B^n(C_1) & \longrightarrow & C_2^n / B^n(C_2) & \longrightarrow & C_3^n / B^n(C_3) & \longrightarrow & 0 \\
\| & & \| & & \| & & \\
\mathrm{Coker}(d_1^{n-1}) & & \mathrm{Coker}(d_2^{n-1}) & & \mathrm{Coker}(d_3^{n-1}) & &
\end{array}$$

As $B^n(C_\ell) \subset \operatorname{Ker}(d_\ell^n)$ $(\ell = 1, 2, 3)$ we get a commutative diagram with exact rows:

$$
\begin{tikzcd}
 & C_1^n / B^n(C_1) \arrow[r] \arrow[d, "f"] & C_2^n / B^n(C_2) \arrow[r] \arrow[d, "g"] & C_3^n / B^n(C_3) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & Z^{n+1}(C_1) \arrow[r] & Z^{n+1}(C_2) \arrow[r] & Z^{n+1}(C_3) &
\end{tikzcd}
$$

Easy check: Kernels of vertical maps are $H^n(C_\ell)$

Cokernels of vertical maps are $H^{n+1}(C_\ell)$

Hence Snake lemma again implies the exactness of the long sequence:

$$
\begin{tikzcd}
\cdots & H^n(C_1) \arrow[r] & H^n(C_2) \arrow[r] & H^n(C_3) \arrow[dll, "\delta"] \\
 & H^{n+1}(C_1) \arrow[r] & H^{n+1}(C_2) \arrow[r] & H^{n+1}(C_3)
\end{tikzcd}
$$

which is (i) of Theorem (14.3).

(15.3) Naturality of connecting homomorphism (part (ii) of Theorem 14.3).

Now assume that we are given two short exact sequences and a commutative diagram between them.

$$\begin{array}{ccccccccc} 0 & \to & C_1^\bullet & \longrightarrow & C_2^\bullet & \longrightarrow & C_3^\bullet & \longrightarrow & 0 \\ & & f\downarrow & & \downarrow g & & \downarrow h & & \\ 0 & \to & D_1^\bullet & \longrightarrow & D_2^\bullet & \longrightarrow & D_3^\bullet & \longrightarrow & 0 \end{array}$$

, and we want to prove the commutativity of

$$\begin{array}{ccc} H^n(C_3) & \xrightarrow{\delta_C^n} & H^{n+1}(C_1) \\ H^n(h)\downarrow & & \downarrow H^{n+1}(f) \\ H^n(D_3) & \xrightarrow{\delta_D^n} & H^{n+1}(D_1) \end{array}$$

.

We will use the diagram of page 4 – defn. of δ via Snake lemma. The commutativity of $H^{n+1}(f)\,\delta_C^n \overset{?}{=} \delta_D^n\, H^n(h)$ follows from that of each smaller piece : (see big cartoon on next page)

$$\begin{array}{ccc} H^n(C_3) & \to & C_3^n/B^n(C_3) \\ \downarrow & & \downarrow \\ H^n(D_3) & \to & D_3^n/B^n(D_3) \end{array} \quad ; \quad \begin{array}{ccc} C_2^n/B^n(C_2) & \to & C_3^n/B^n(C_3) \\ \downarrow & & \downarrow \\ D_2^n/B^n(D_2) & \to & D_3^n/B^n(D_3) \end{array} \quad ;$$

$$\begin{array}{ccc} C_2^n/B^n(C_2) & \to & Z^{n+1}(C_2) \\ \downarrow & & \downarrow \\ D_2^n/B^n(D_2) & \to & Z^{n+1}(D_2) \end{array} \quad ; \quad \begin{array}{ccccc} H^{n+1}(C_1) & \leftarrow & Z^{n+1}(C_1) & \to & Z^{n+1}(C_2) \\ \downarrow & & \downarrow & & \downarrow \\ H^{n+1}(D_1) & \leftarrow & Z^{n+1}(D_1) & \to & Z^{n+1}(D_2) \end{array}$$

$$
\begin{tikzcd}
 & & H^n(C_3) \arrow[rrr] \arrow[d] & & & H^n(D_3) \arrow[d] \\
 & C_2^n / B^n(C_2) \arrow[r] \arrow[d] \arrow[rrr, dashed, bend left] & C_3^n / B^n(C_3) \arrow[rrr, dashed, bend left] & & D_2^n / B^n(D_2) \arrow[r] \arrow[d] & D_3^n / B^n(D_3) \\
Z^{n+1}(C_1) \arrow[r] \arrow[d] \arrow[rrr, dashed, bend left] & Z^{n+1}(C_2) \arrow[rrr, dashed, bend left] & & Z^{n+1}(D_1) \arrow[r] \arrow[d] & Z^{n+1}(D_2) & \\
H^{n+1}(C_1) \arrow[rrr] & & & H^{n+1}(D_1) & &
\end{tikzcd}
$$

Dessin d'enfants

(15.4) Homotopy. — Let $C^\bullet$ and $D^\bullet$ be two cochain complexes and let $\alpha^\bullet : C^\bullet \longrightarrow D^\bullet$ be a morphism. We say α is <u>null-homotopic</u> <u>$(\alpha \sim 0)$</u> if $\exists\ s^n : C^n \to D^{n-1}$ $(\forall\, n \in \mathbb{Z})$

s.t. $\alpha = ds + sd$ (w/ appropriate superscripts)

$$\begin{tikzcd}
 & C^n \arrow[r, "d_C^n"] \arrow[dl, "s^n"'] \arrow[d, "\alpha^n"] & C^{n+1} \arrow[dl, "s^{n+1}"] \\
D^{n-1} \arrow[r, "d_D^{n-1}"'] & D^n &
\end{tikzcd}$$

As usual, if $\beta, \gamma \in \mathrm{Hom}_{\mathbb{K}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$, we say <u>$\beta \sim \gamma$</u> ($\beta$ is homotopic to γ) if $\beta - \gamma$ is null-homotopic (i.e. we can find $s^n : C^n \longrightarrow D^{n-1}$ for every $n \in \mathbb{Z}$ s.t. $\beta - \gamma = ds + sd$).

Proposition. — (i) $\mathrm{Hom}^{\mathrm{null}}_{\mathbb{K}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$ [null-homotopic maps] is an abelian subgroup of $\mathrm{Hom}_{\mathbb{K}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$

(ii) $\alpha^\bullet \sim 0$ $(\alpha^\bullet : C^\bullet \to D^\bullet)$ $\Rightarrow$ $\beta\alpha \sim 0$ $\forall\, \beta : D^\bullet \to E^\bullet$ and $\alpha\gamma \sim 0$ $\forall\, \gamma : A^\bullet \to C^\bullet$

[analogue of "ideals in rings" – good for moding out!]

(iii) $\alpha^{\bullet} \sim 0 \Rightarrow H^n(\alpha^{\bullet}) = 0 \quad \forall n \in \mathbb{Z}$.

Proof. – (i) $\alpha \sim 0 \Rightarrow \exists\, s^n : C^n \longrightarrow D^{n-1}$ such that $\alpha = ds + sd$.

Then $(-\alpha) = d(-s) + (-s)d \quad \Rightarrow \quad -\alpha \sim 0$.

$\alpha + \alpha' = d(s+s') + (s+s')d \quad \Rightarrow \quad \alpha + \alpha' \sim 0$.

(ii) Assume $C \xrightarrow{\alpha} D \xrightarrow{\beta} E$ & $\alpha \sim 0$ (i.e. $\alpha = ds + sd$)

Then $\beta\alpha = \beta(ds + sd) = d(\beta s) + (\beta s)d$ ($\beta d = d\beta$ by defn.)

$\Rightarrow \beta\alpha \sim 0$.

Same argument for $\alpha\gamma \sim 0$.

(iii) Again if $\alpha : C \longrightarrow D$ and $\alpha \sim 0$ (i.e. $\alpha = ds + sd$ for some s)

$$\begin{tikzcd} & C^n \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, "\alpha"] & C^{n+1} \arrow[dl, "s"] \\ D^{n-1} \arrow[r, "d"'] & D^n & \end{tikzcd}$$

$\forall\, x \in \mathrm{Ker}(d_C^n)$

$\alpha(x) = ds(x) + s\underbrace{\cancel{d}(x)}_{0}$

$\in$ Image of (d_D^{n-1})

$\Rightarrow H^n(\alpha) = 0$. $\square$