Lecture 16

Recall that last time we introduced the notion of homotopy of two morphisms $\alpha, \beta : C^{\cdot} \longrightarrow D^{\cdot}$. The terminology comes from algebraic topology: X : a topological space $\rightsquigarrow$ $C_{\cdot}(X)$ (singular chain complex)

$X \xrightarrow{f} Y$ cnts $\rightsquigarrow$ $f_* : C_{\cdot}(X) \longrightarrow C_{\cdot}(Y)$

$f, g : X \to Y$ homotopic $\rightsquigarrow$ f_* and g_* are homotopic

Today we will prove analogous assertions in homological algebra.

(16.1) Again, let R be a commutative ring with $1 \neq 0$. Let $\mathcal{A} = R\text{-mod}$. Recall $Q \in \mathcal{A}$ (resp. $P \in \mathcal{A}$) is called injective (resp. projective) if any of the following equivalent conditions hold:

(1) $\mathrm{Hom}_{\mathcal{A}}(-, Q)$ is exact (resp. $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact).
(see Defn. 10.3 page 5)

(2) $\forall$ injective morphism $Q \hookrightarrow \widetilde{Q}$ (resp. surjective morphism $\widetilde{P} \twoheadrightarrow P$), we have $\widetilde{Q} = Q \oplus \overline{Q}$ (resp. $\widetilde{P} = P \oplus \overline{P}$).

(3) $\forall$ injective morphism $0 \to A \xrightarrow{i} B$ and arbitrary $A \xrightarrow{f} Q$, $\exists$ $\widetilde{f} : B \to Q$ s.t. $\widetilde{f} i = f$

(resp. $\forall$ surjective morphism $B \xrightarrow{\pi} C \to 0$ and any $g : P \to C$, $\exists\ \tilde{g} : P \to B$ such that $\pi \tilde{g} = g$)

[see § 11.4, 11.5 (page 8) and Theorem 12.1 page 1].

(16.2) Definition. — An <u>injective resolution</u> of $M \in \mathcal{A}$ is a <u>cochain</u> complex $0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots$ $(\in K^{\cdot}(\mathcal{A}))$

s.t.
$$\begin{cases} \bullet\ H^n(I^{\cdot}) = 0 \quad \forall\, n \geq 1 & \text{(i.e. it is exact at } I^n\ \forall\, n \geq 1) \\ \bullet\ I^k \text{ is injective } \quad \forall\, k \geq 0 \\ \bullet\ \mathrm{Ker}(d^0) = M = H^0(I^{\cdot}) \end{cases}$$

(in other words, $0 \to M \xrightarrow{i} I^0 \to I^1 \to \cdots$ is <u>exact</u>)

A <u>projective</u> (resp. other adjectives like flat, free etc.) <u>resolution</u> of $N \in \mathcal{A}$ is a <u>chain</u> complex $(\in K_{\cdot}(\mathcal{A}))$

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

s.t.
$$\begin{cases} \bullet\ H_n(P_{\cdot}) = 0 \quad \forall\, n \geq 1 & \text{(i.e. exact at } P_n\ \forall\, n \geq 1) \\ \bullet\ P_k \text{ is projective } \quad \forall\, k \geq 0 \\ \bullet\ H_0(P_{\cdot}) = P_0 / \mathrm{Im}(d_1) = N \end{cases}$$

(i.e. $\cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\pi} N \longrightarrow 0$ is exact)

Remarks. – (i) Injective / projective resolutions are not unique, but only up to homotopy (Today's main thm.)

(ii) In practice, projective resolutions (or free resolutions – i.e. each P_k is a free R-module, i.e. $P_k = R^{\oplus(I)} = \bigoplus_{i \in I} R$) are computationally friendlier than injective resolutions.

(iii) For example, (proof next week)

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{d^0} \mathbb{Q}/\mathbb{Z} \to 0 \quad \text{is an injective}$$

resolution of $\mathbb{Z}$ in $\mathcal{A} = \mathbb{Z}\text{-mod} = \mathcal{A}b$.

(iv) Let $R = K[x,y]$ and let $N = (x,y) \overset{\text{ideal}}{\subset} R$.

$$\begin{array}{ccccccc} & & 1 \longmapsto (y,-x) & & & & \\ 0 \to & R & \xrightarrow{d_0} & R \oplus R & \xrightarrow{\pi} & N & \to 0 \\ & & & (1,0) & \longmapsto & x & \\ & & & (0,1) & \longmapsto & y & \end{array}$$

is a free resolution

"relations" "generators".

(16.3) Theorem. – Let $M, M' \in \mathcal{A}$. Assume we are given two ~~exact sequences~~ injective resolutions.

$$0 \to M \xrightarrow{i} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$$

$$0 \to M' \xrightarrow{j} J^0 \xrightarrow{d^0} J^1 \xrightarrow{d'} J^2 \to \cdots$$

Then for every R-linear map $f : M \longrightarrow M'$, there exists $f^{\bullet} : I^{\bullet} \longrightarrow J^{\bullet}$, unique up to homotopy, s.t. $f^{0} \circ i = j \circ f$.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & I^0 \arrow[r] \arrow[d, "f^0"] & I^1 \arrow[r] \arrow[d, "f^1"] & \cdots \\
0 \arrow[r] & M' \arrow[r, "j"'] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots
\end{tikzcd}$$

Similary if we are given two projective resolutions (of N & $N' \in \mathcal{A}$) $P_{\bullet}$ and $P'_{\bullet}$, then $\forall$ ~~f~~ $g : N \longrightarrow N'$, $\exists$ unique up to homotopy $g_{\bullet} : P_{\bullet} \longrightarrow P'_{\bullet}$ s.t.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "g_2"'] & P_1 \arrow[r] \arrow[d, "g_1"'] & P_0 \arrow[r] \arrow[d, "g_0"'] \arrow[dr, phantom, "\circlearrowright"] & N \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & N' \arrow[r] & 0
\end{tikzcd}$$

(16.4) As a consequence of Theorem 16.3 above, we get that:

Corollary. Injective resolutions are unique up to homotopy.
(Projective)

Pf. Let $I_1^{\bullet}$ and $I_2^{\bullet}$ be two injective resolutions of $M \in \mathcal{A}$. Take $f = \mathrm{Id}_M$ in the theorem to get (unique up to homotopy) $\alpha^{\bullet} : I_1^{\bullet} \longrightarrow I_2^{\bullet}$ and $\beta : I_2^{\bullet} \longrightarrow I_1^{\bullet}$. Then $\beta\alpha : I_1 \longrightarrow I_1$

Lifts the identity map $M \xrightarrow{Id_M} M$. That is,

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & I_1^0 \arrow[r] \arrow[d, "(\beta\alpha)^0"] & I_1^1 \arrow[r] \arrow[d, "(\beta\alpha)^1"] & I_1^2 \arrow[r] \arrow[d, "(\beta\alpha)^2"] & \cdots \\ 0 \arrow[r] & M \arrow[r] & I_1^0 \arrow[r] & I_1^1 \arrow[r] & I_1^2 \arrow[r] & \cdots \end{tikzcd}$$

But so does $Id_{I_1^\cdot}$. By uniqueness up to homotopy part of the theorem $\beta\alpha \sim Id_{I_1^\cdot}$. Similarly $\alpha\beta \sim Id_{I_2^\cdot}$. Thus $I_1^\cdot$ and $I_2^\cdot$ are homotopic. The same argument works for projective resolutions. □

(16.5) Proof of Theorem 16.3 - I. — Existence of $f^\cdot$.

Base Case. - f^0.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\text{injective}"] \arrow[d, "j\circ f"'] & I^0 \arrow[dl, dashed, "\text{exists, since } J^0 \text{ is injective } R\text{-mod.} =: f^0"] \\ & J^0 & \end{tikzcd}$$

Now assume $f^0, f^1, \ldots, f^n$ $(n \geq 0)$ have been constructed so that (set $f^{-1} = f : M \to M'$)

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"] & I^0 \arrow[r] \arrow[d, "f^0"] & \cdots \arrow[r] & I^n \arrow[d, "f^n"] \\ 0 \arrow[r] & M' \arrow[r, "j"'] & J^0 \arrow[r] & \cdots \arrow[r] & J^n \end{tikzcd}$$

($\downarrow f^1 \cdots \downarrow f^{n-1}$) so all squares commute.

Define $I^n / Ker(d_I^n) \longrightarrow J^{n+1}$ via :

$$\begin{tikzcd}
I^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & I^n \arrow[r] \arrow[d, "f^n"] & I^n/\mathrm{Im}(d^{n-1}) \arrow[d, "\overline{f^n}"] \arrow[dr, "\text{defn. } \cancel{\text{of}}\ d^n_J \circ \overline{f^n}"] & \\
J^{n-1} \arrow[r, "d^{n-1}"'] & J^n \arrow[r] & J^n/\mathrm{Im}(d^{n-1}) \arrow[r, "d^n_J"'] & J^{n+1}
\end{tikzcd}$$

(factors through $J^n/\mathrm{Im}(d^{n-1})$ because $\mathrm{Im}(d^{n-1}) \subset \mathrm{Ker}(d^n)$ in fact equal)

Now

$$\begin{tikzcd}
I^n/\mathrm{Im}(d^{n-1}) = I^n/\mathrm{Ker}(d^n) \arrow[r, hook] \arrow[d, "d^n_J \circ \overline{f^n}"'] & I^{n+1} \arrow[dl, dashed, "\text{defn of } f^{n+1}\text{; exists because } J^{n+1} \text{ is injective.}"] \\
J^{n+1} &
\end{tikzcd}$$

Remark. — We didn't use <u>injectivity of objects I^k</u> $(k \geq 0)$.

We didn't use exactness of $J^{\bullet}$, but only that it is a complex.

(16.6) Proof of Thm 16.3 part II. — Uniqueness.

If $f^{\bullet}$ and $g^{\bullet}$ are two morphisms that lift the same $f : M \to M'$, their difference lifts $0 : M \to M'$. Thus, it suffices to prove that if $f^{\bullet}$ lifts zero morphism, then it is homotopic to 0 (i.e. null homotopic). Thus we need to construct $s^n : I^n \longrightarrow J^{n-1}$ s.t. $f^\bullet = ds + sd$.

Base case. $s^0 : I^0 \longrightarrow J^{-1} = M'$, ~~so~~ $s^0 := 0$.

Now assume that we successfully constructed $s^0, \dots, s^n$ $(n \geq 0)$

$$\begin{tikzcd}
0 \arrow[r] & M = I^{-1} \arrow[r] \arrow[d, "f^{-1}=0"'] & I^0 \arrow[r] \arrow[d, "f^0"] \arrow[dl, "0=s^0"'] & I^1 \arrow[r] \arrow[d, "f^1"] \arrow[dl, "s^1"'] & \cdots \arrow[r] & I^n \arrow[dl, "s^n"'] \\
0 \arrow[r] & M' = J^{-1} \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots \arrow[r] & J^n
\end{tikzcd}$$

So that $f^k = d_J^{k-1} s^k + s^{k+1} d_I^k \quad \forall \underbrace{0 \leq k \leq n-1}$

(empty condition if $n=0$)

Focus on

$$\begin{tikzcd}
& I^{n-1} \arrow[r, "d_I^{n-1}"] \arrow[dl, "s^{n-1}"'] \arrow[d, "f^{n-1}"] & I^n \arrow[r, "d_I^n"] \arrow[dl, "s^n"'] \arrow[d, "f^n"] & I^{n+1} \\
J^{n-2} \arrow[r, "d_J^{n-2}"'] & J^{n-1} \arrow[r, "d_J^{n-1}"'] & J^n &
\end{tikzcd}$$

[obvious conventions: $s^{-1} = 0$; $J^{-2} = 0$]

Set $g = f^n - d_J^{n-1} s^n : I^n \longrightarrow J^n$

Now $g \circ d_I^{n-1} = f^n d_I^{n-1} - d_J^{n-1} s^n d_I^{n-1}$

$= d_J^{n-1} \left(f^{n-1} - s^n d^{n-1} \right) = d^{n-1} d^{n-2} s^{n-1} = 0$

$\Rightarrow$ g factors through

$$\begin{tikzcd}
I^n / \mathrm{Im}(d^{n-1}) = I^n / \mathrm{Ker}(d^n) \arrow[r, hook] \arrow[d, "g"] & I^{n+1} \arrow[dl, dashed, "\text{as } J^n \text{ is injective}"] \\
J^n &
\end{tikzcd}$$

$\square$