Lecture 17

(17.0) Recall that we are working with $\mathcal{A}$ = R-mod the abelian category of R-modules, where R is a commutative ring with $1 \neq 0$. The aim of today's lecture is to prove that injective resolutions exist. After each result we will describe the minimal list of properties of R-mod used in the proof and refer to Appendix (Optional reading) A, for the general results.

(17.1) Baer's Criterion for injectivity.

Lemma. Let $M \in \mathcal{A}$. For M to be injective, it is necessary and sufficient that for every ideal $\mathfrak{a} \subset R$ and an R-linear map $\mathfrak{a} \xrightarrow{f} M$, there exists $g : R \to M$ s.t. $g|_{\mathfrak{a}} = f$. [Note: g is completely determined by $m_g = g(1)$; so, we are saying $f(a) = a.m$; for some $m \in M$]

Proof ($\Rightarrow$) by defn. of injective module,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\forall"] \arrow[d, "\forall"'] & B \arrow[dl, dotted, "\exists"] \\ & M & \end{tikzcd}$$

Take $A = \mathfrak{a} \hookrightarrow R = B$.

($\Leftarrow$) Assume we are given an injective R-linear map $0 \to A \xrightarrow{i} B$ and an arbitrary R-linear map $A \xrightarrow{f} M$. We view injective morphisms as sub-R-modules of B, after taking their images.

sub-R-module of B containing A

$$\mathcal{P} := \left\{ (f', A') \;\middle|\; \begin{array}{l} A \subset A' \subset B \\ f' : A' \longrightarrow M \text{ is an } R\text{-linear map} \\ \text{s.t. } f'|_A = f \end{array} \right\}$$

$\mathcal{P}$ is a partially ordered set : $(f_1, A_1) \leq (f_2, A_2)$ if

$A_1 \subset A_2$ and $f_2|_{A_1} = f_1$.

If $(f_1, A_1) \leq (f_2, A_2) \leq \cdots$ is a totally ordered chain in $\mathcal{P}$, then $\tilde{A} = \bigcup_{j=1}^{\infty} A_j \subset B$ with $\tilde{f} : \tilde{A} \longrightarrow M$ given by

$a \in \tilde{A} \Rightarrow \exists\, j$ s.t. $a \in A_j$. $\tilde{f}(a) := f_j(a)$,

is the max. of $\{(f_i, A_i)\}$. By Zorn's lemma, we can find a max'l $(g, C) \in \mathcal{P}$. We want to prove that $C = B$.

If $C \neq B$, then for an $x \in B \setminus C$, let

R-linear

$\mathfrak{a} := \{ r \in R \mid rx \in C \}$ and define
$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\varphi} & M \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ r & \longmapsto & g(rx) \end{array}$$

By our hypothesis, $\exists\, m \in M$ s.t. $\varphi(r) = r \cdot m$

Set $\tilde{C} = C + Rx$ (R-submod. of B generated by C and x). $\tilde{g} : \tilde{C} \longrightarrow M$ is defined by

$$\tilde{g}(c + rx) = g(c) + r.m$$ ($m \in M$ from the last paragraph).

While it is true that every element of $\tilde{C}$ is of the form $c + r.x$ for some $c \in C$ and $r \in R$; this expression need not be unique. Thus we must check that $\tilde{g}$ is well-defined, i.e.

$$c_0 + r_0 x = 0 \quad \Rightarrow \quad g(c) + r.m = 0$$ (To show)

$c + rx = 0$ means $r \in \mathfrak{a}$. By our construction of $m \in M$, this means $r.m = \varphi(r) = g(r \cdot x) = g(-c) = -g(c)$

$$\Rightarrow \quad g(c) + r.m = 0$$ as required.

Thus
$$\begin{tikzcd} C \arrow[r, phantom, "\subsetneq"] \arrow[d, "g"'] & \tilde{C} \arrow[dl, "\tilde{g}"] \\ M & \end{tikzcd}$$
contradicts maximality of (g, C) and we are done. $\square$

Remark (not part of Alg II). — List of properties needed from R-mod, to make the proof work.
(see Appendix A)

(i) $\varinjlim$ exist and is left exact

(ii) Existence of a "generator" ($R \in R$-mod in our case).

Definition. — Let $\mathcal{C}$ be an abelian category. An object $\mathcal{U} \in \mathcal{C}$ is called <u>generator of $\mathcal{C}$</u> if for every injective $A \overset{i}{\hookrightarrow} B$ which is NOT an isomorphism, $\exists\ f : \mathcal{U} \to B$ which does not factor through i (i.e. there is <u>no</u> $\bar{f} : \mathcal{U} \to A$ s.t. $f = i\bar{f}$). In other words,
$\mathrm{Hom}(\mathcal{U}, A) \xrightarrow[i\circ -]{} \mathrm{Hom}(\mathcal{U}, B)$ is <u>NOT</u> a bijection.)

(17.2) Corollaries of Lemma 17.1.

Assume R is an integral domain (i.e., $r_1 r_2 = 0 \Rightarrow r_1 = 0$ or $r_2 = 0$).

Definition. An R-module M is said to be <u>divisible</u> if

$\forall\ r \in R$, $(r \neq 0)$. $r_M : M \longrightarrow M$, $x \longmapsto r.x$ is surjective.

Corollary (of Lemma 17.1). Assume R is an integral domain.

(1) $Q \in R$-mod is injective $\Rightarrow$ Q is divisible.

(2) $M \in R$-mod, torsion free & divisible $\Rightarrow$ M is injective

(3) If in addition R is a Principal Ideal Domain (i.e. every ideal in R is generated by one element), then

injective $\iff$ divisible

Proof. (1) Let Q be an injective R module and $r \in R$, $r \neq 0$.

We want to show that $r_Q : Q \to Q$ is surjective.

Since $r \neq 0$ and R is integral domain; $r_R : R \to R$, $s \mapsto rs$ is an injective morphism of R-modules. Thus, by defn. of injective module

$$\mathrm{Hom}_R(R,Q) \xrightarrow{-\circ r_R} \mathrm{Hom}_R(R,Q) \quad \text{is surjective.}$$

Now $\mathrm{Hom}_R(R,Q) \simeq Q$ and via this identification, $-\circ r_R$ equals

$$\begin{array}{ccc} \eta & \longmapsto & \eta(1) \\ \in & & \in \\ \mathrm{Hom}_R(R,Q) & & Q \end{array}$$

$$\begin{tikzcd}
\eta \in \mathrm{Hom}_R(R,Q) \arrow[r, "-\circ r_R"] \arrow[d, "\wr"'] & \mathrm{Hom}_R(R,Q) \ni \eta\circ r_R \arrow[d, "\wr"] \\
\eta(1) \in Q \arrow[r, "r_Q"'] & Q \ni \eta(r) = r\cdot\eta(1)
\end{tikzcd}$$

and we are done.

(2) and (3) are proved as follows. ((⇒) of (3) follows from (1)).

Let M be a divisible R-module. Our hypothesis is either

- R is a PID,

or

- M is torsion-free (& R is an integral domain)

To show: M is injective.

By Lemma 17.1, it is enough to prove that $\forall$ (non-zero) ideal $\mathfrak{a} \subset R$; and an R-linear map $f : \mathfrak{a} \to M$;

there exists $m \in M$ such that $f(a) = a.m$.

So choose $x \in \mathfrak{a}$ s.t. $\begin{cases} \text{either} \cdot \mathfrak{a} = (x) \text{ if } R \text{ is PID} \\ \text{or just} \cdot x \neq 0 \text{ if } M \text{ is torsion-free.} \end{cases}$

As M is divisible $M \xrightarrow{\text{mult by } x} M$ is surjective, i.e. $\exists\, m \in M$

$$(\exists\, m \in M) \longrightarrow f(x)$$

s.t. $x \cdot m = f(x)$. <u>Claim</u>. $\forall\, a \in \mathfrak{a}$, $f(a) = a.m$.

The claim is clearly true for $a \in (x)$; i.e. $a = r.x$ for some $r \in R$, because then $f(a) = f(rx) \underset{\substack{\uparrow \\ (f \text{ is } R\text{-linear})}}{=} r f(x) = r.x.m = a.m.$

This finishes the proof in the case of R being a PID. For the other case, we have, for any $a \in \mathfrak{a}$, $ax \in (x)$, so

$$f(ax) = ax.m = x.(a.m)$$

$$f(ax) = f(xa) = x.(f(a))$$

$\Rightarrow$ $x.(a.m - f(a)) = 0$. Now $x \neq 0$ and M is torsion free, implies

$$a.m = f(a) \quad \text{as required.}$$

$\square$

(17.3) Examples. 1. R is integral domain, $K = Q(R) =$ field of fractions of R is an injective R module.

2. $R = \mathbb{Z}$, $M = \mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ is injective.

3. (Homework). R : PID, $x \in R \setminus \{0\}$;
$R/(x)$ is an injective $R/(x)$ - module.

4. $R =$ field. Every R-module (i.e. vector space) is injective.

(17.4) Constructing more examples.

Proposition. – Let $Q \in \underline{Ab}$ be an injective abelian group. Then

$$I_Q := \operatorname{Hom}_{\underline{Ab}}(R, Q) \quad \text{with } R\text{-module structure}:$$

$$(r \cdot f)(s) = f(rs) \qquad \forall\, r, s \in R;\ f \in \operatorname{Hom}_{\underline{Ab}}(R, Q)$$

is an injective R-module.

Proof. To prove (as a consequence of Lemma 17.1)

$$\forall\, \mathfrak{a} \subset R \text{ ideal} \quad \text{and} \quad f : \mathfrak{a} \longrightarrow I_Q \quad (R\text{-linear})$$

$$\exists\, \psi \in I_Q \text{ s.t. } f(a) = a \cdot \psi$$

Homework.

$$\begin{array}{ccc} \operatorname{Hom}_R(\mathfrak{a}, \operatorname{Hom}_{\mathbb{Z}}(R, Q)) & \xrightarrow{\sim} & \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{a}, Q) \\ \Psi & & \Psi \\ f & \longmapsto & \underbrace{\{a \in \mathfrak{a} \mapsto f(a)(1)\}}_{=: \rho_f} \end{array}$$

As $\rho_f : \mathfrak{a} \longrightarrow Q$ (gp. hom) and $\mathfrak{a} \hookrightarrow R$; by injectivity of Q:

$$\begin{tikzcd} \mathfrak{a} \arrow[r, hook] \arrow[d, "\rho_f"'] & R \arrow[dl, dashed, "\text{defn. of } \psi"] \\ Q & \end{tikzcd}$$

.

To prove: $\forall\ a \in \mathfrak{A},\ \underbrace{f(a) = a\cdot\psi}$ (both sides are $\mathbb{Z}$-linear maps $R \to Q$)

evaluate at $1 \in R$ to get

$$\left.\begin{array}{l} \text{LHS} = \rho_f(a) \\ \text{RHS} = \psi(a) \end{array}\right\} \text{equal by defn of } \psi.$$

$\square$