

(18.0) Recall that last time we proved:

- (a) To prove that an R -module is injective, it is enough to extend R -linear maps from ideals of R ($\mathcal{O} \rightarrow M$) to $(R \rightarrow M)$. (Lemma 17.1)
- (b) If Q is an injective abelian group (such as \mathbb{Q} or \mathbb{Q}/\mathbb{Z}) then $I_Q := \text{Hom}_{\mathbb{Z}}(R, Q)$ is an injective R -module.

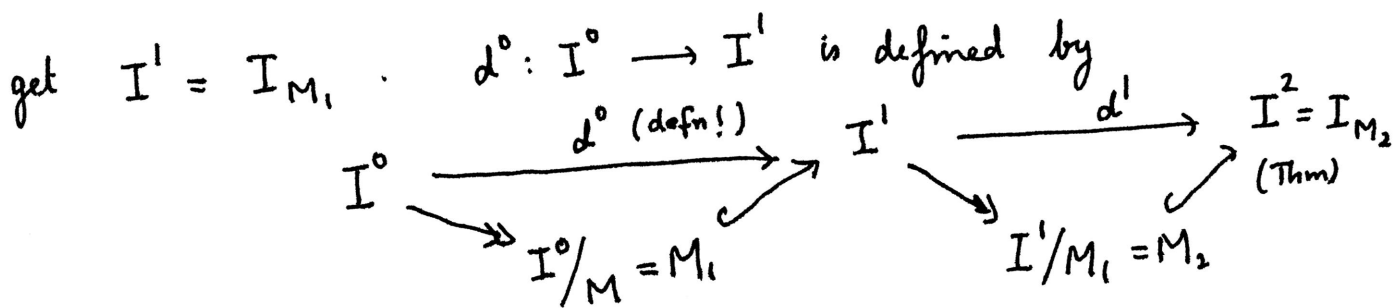
(18.1) Theorem. - The category $\mathcal{A} = R\text{-mod}$ has enough injective. That is, $\forall M \in \mathcal{A}$, there exists an injective morphism $M \hookrightarrow I_M$ where I_M is an injective R module.

Cor Injective resolutions exist. That is, $\forall M \in \mathcal{A}$, there exists a complex $0 \rightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots$ s.t.

- $\text{Ker}(d^n) = \text{Im}(d^{n-1}) \quad \forall n \geq 1$ (exactness at $I^n \quad \forall n \geq 1$)
- each I^k is injective ($k \geq 0$)
- $\text{Ker}(d^0) = M$.

Pf. of Cor. Set $I^0 = I_M$ from the theorem.

$M \hookrightarrow I^0 \rightarrow I^0/M =: M_1$. Apply the theorem to M_1 , and



and continue. □

(18.2) Proof of Thm 18.1. -

Step 1. - An R module, E , is said to be a cogenerator if
 $\forall M \in R\text{-mod}$ and $m \in M, m \neq 0, \exists \varphi: M \rightarrow E$ s.t. $\varphi(m) \neq 0$.
 (R -linear)

Lemma. - \mathbb{Q}/\mathbb{Z} is a cogenerator in Ab. (= \mathbb{Z} -mod).

Proof. - Let A be an abelian group and $a \in A, a \neq 0$.

Consider the subgroup generated by a , say $A_1 \hookrightarrow A$.
 Depending on whether a is of finite order or not, we have
 two cases. - (i) $A_1 \cong \mathbb{Z}$ (ii) $A_1 \cong \mathbb{Z}/N\mathbb{Z}$
 $a \leftrightarrow 1$ $a \leftrightarrow 1$

In any case there is a non-zero map (sending a to a non-zero element $\frac{1}{N} \in \mathbb{Q}/\mathbb{Z}$)

$$\begin{array}{ccccc} A_1 & \longrightarrow & \mathbb{Z}/N\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ a & \longmapsto & 1 & \longmapsto & \frac{1}{N} \end{array}$$

As \mathbb{Q}/\mathbb{Z} is injective, we can lift this to

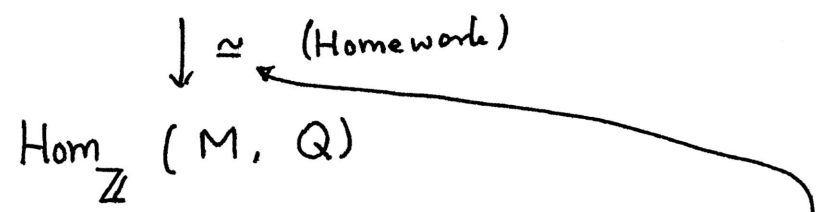
$$\begin{array}{ccc} A_1 & \hookrightarrow & A \\ \downarrow & \nearrow \exists & \\ \mathbb{Q}/\mathbb{Z} & & \end{array}$$

and we are done. □

Step 2. If $Q \in \underline{Ab}$ is an injective cogenerator, then
 so is $I_Q = \text{Hom}_{\mathbb{Z}}(R, Q) \in R\text{-mod}$.

Proof. - Given $M \in R\text{-mod}$ and $m \in M$ ($m \neq 0$)

we need to find $f \in \text{Hom}_R(M, \text{Hom}_{\mathbb{Z}}(R, \mathbb{Q}))$ s.t. $f(m) \neq 0$



Pick $\rho \in \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q})$ s.t. $\rho(m) \neq 0$. Let f be its preimage under

i.e. $f(x)(r) = \rho(r \cdot x)$. Then $f(m): R \rightarrow \mathbb{Q}$

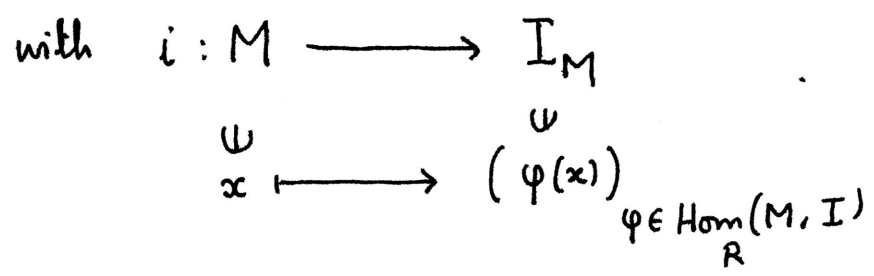
\mathbb{Z} -linear

is a non-zero map since $f(m)(1) = \rho(m) \neq 0$.

Step 3. Direct product of injective modules is injective. [Homework].

Step 4. Let $I \in R\text{-mod}$ be an injective cogenerator. (exists by Steps 1 & 2)

For any $M \in R\text{-mod}$ define $I_M = \prod_{\varphi \in \text{Hom}_R(M, I)} I_{(\varphi)}$ $\left(\begin{array}{l} I_{(\varphi)} = I \\ \forall \varphi \end{array} \right)$



This is clearly an R -linear map. It is injective because $\forall x \in M, x \neq 0$,

there exists $\psi: M \rightarrow I$ s.t. $\psi(x) \neq 0$; i.e. ψ^{th} component of $i(x)$ is non-zero, hence $i(x) \neq 0$.

□

(18.3) Projective / Free resolutions. -

Lemma. - For every set I , the direct sum $R^{(I)} (= \bigoplus_{i \in I} R_{(i)}$ each $R_{(i)} = R$ as an R -module) is projective. (such R -modules are called free).

Proof. We need to prove that $\text{Hom}_R(R^{(I)}, -)$ is exact. By defn. (of direct sum) $\text{Hom}_R(R^{(I)}, X) = \prod_{i \in I} \text{Hom}_R(R, X)$

and $\text{Hom}_R(R, X) \cong X$. Thus, we need to prove that

$$\begin{array}{ccc} \psi & & \psi \\ \eta & \longmapsto & \eta(1) \end{array}$$

for every exact seq. $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ of R -modules and a set I , the sequence $0 \rightarrow X^I \xrightarrow{f^I} Y^I \xrightarrow{g^I} Z^I \rightarrow 0$ is exact (notation: $M^I = \prod_{i \in I} M$). This is an easy exercise. (Problem 7 HW4). \square

(18.4) Proposition. - $\forall M \in R\text{-mod}$, there exists a free object F and a surjective R -linear map $F \xrightarrow{\pi} M$.

Proof. Take $F = R^{(M)}$ $\xrightarrow{\pi}$ M induced from

$$\begin{array}{ccc} R & = & R \longrightarrow M \\ \uparrow & \psi & \downarrow \\ m^{\text{th}} & & m \\ \text{copy} & & \end{array}$$

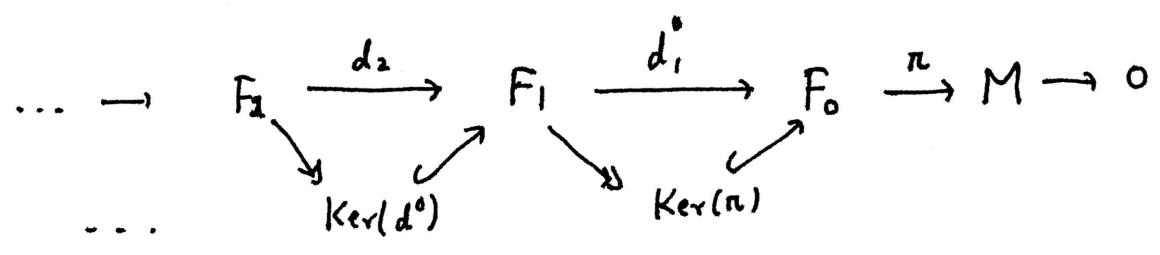
(It will suffice to take any generating set in M)

Cor. Let $P \in R\text{-mod}$ be a projective R -module. Then there exists $P' \in R\text{-mod}$ s.t. $P \oplus P'$ is free.

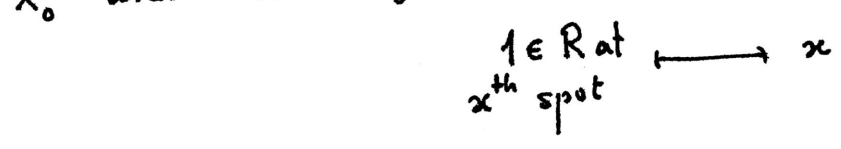
Pf.- Use the proposition to get a surjective map $F \twoheadrightarrow P$; F is free

and then use Thm (12.2) page 2 to see that $F = P \oplus P'$. \square

(18.5) Just as the proof of the corollary of Thm 18.1, we get that every $M \in R\text{-mod}$, admits a (free) resolution; (projective)

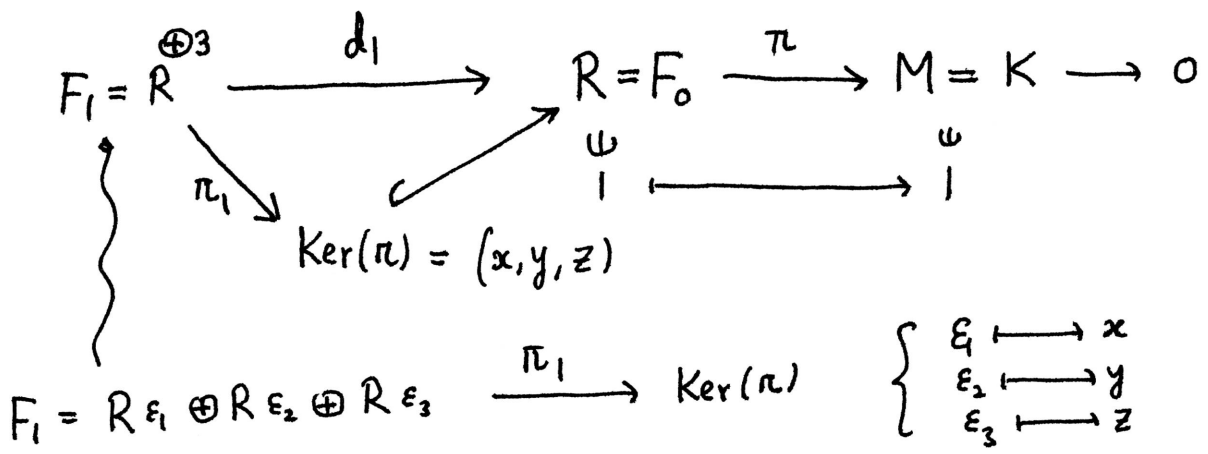


"Algorithmic construction": (1) Pick a set of generators of M , say x_0 and set $F_0 = R^{(x_0)} \xrightarrow{\pi} M$.

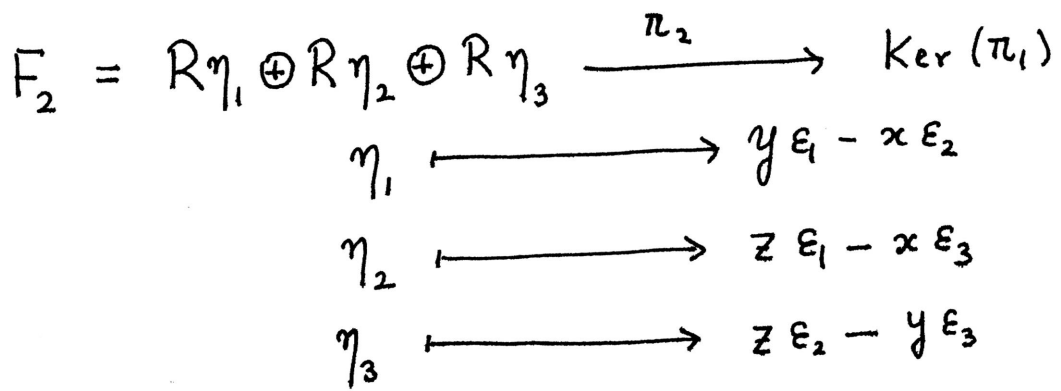


(2) Work out generators of the $\text{Ker}(\pi)$, say X_1 ; $F_1 := R^{(X_1)} \rightarrow \text{Ker}(\pi)$ and continue. (syzygies!)

e.g. $R = K[x, y, z]$ (polynomial ring in 3 var) $M = K$ with x, y, z acting as zero maps.

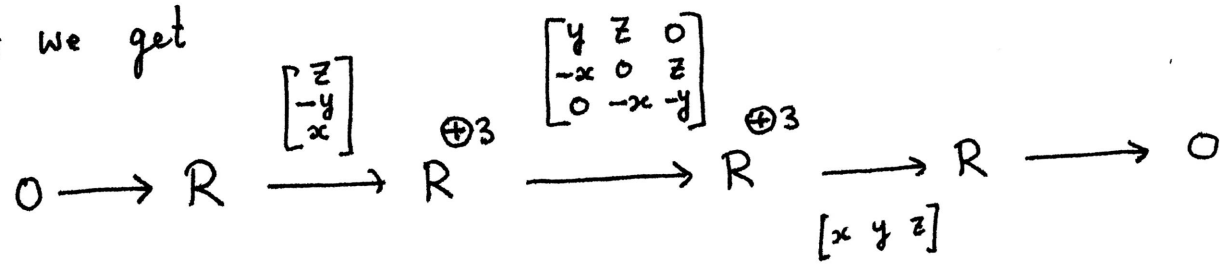


Step 2. $\text{Ker}(d_1) = \text{Ker}(\pi_1)$ contains

$$\begin{aligned}
 & y\varepsilon_1 - x\varepsilon_2 \\
 & z\varepsilon_1 - x\varepsilon_3 \\
 & z\varepsilon_2 - y\varepsilon_3
 \end{aligned}$$


Step 3. $\text{Ker}(\pi_2) = z\eta_1 - y\eta_2 + x\eta_3 \left(\mapsto \begin{array}{l} yz\varepsilon_1 - xz\varepsilon_2 \\ -yz\varepsilon_1 + xy\varepsilon_3 = 0 \\ +xz\varepsilon_2 - xy\varepsilon_3 \end{array} \right)$

Thus we get



Checking exactness is a computation in linear algebra (use elimination to compute the kernel of a matrix.)