Lecture 18

(18.0) Recall that last time we proved:

(a) To prove that an R-module is injective, it is enough to extend R-linear maps from ideals of R $(\mathfrak{a} \to M)$ to $(R \to M)$. (Lemma 17.1)

(b) If Q is an injective abelian group (such as $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$) then $I_Q := \mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is an injective R-module.

(18.1) Theorem. – The category $\mathcal{A} = R$-mod has enough injective. That is, $\forall\, M \in \mathcal{A}$, there exists an injective morphism $M \hookrightarrow I_M$ where I_M is an injective R module.

<u>Cor</u> Injective resolutions exist. That is, $\forall\, M \in \mathcal{A}$, there exists a complex $0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$ s.t.

- $\mathrm{Ker}(d^n) = \mathrm{Im}(d^{n-1})$ $\forall\, n \geq 1$ (exactness at I^n $\forall\, n \geq 1$)
- each I^k is injective $(k \geq 0)$
- $\mathrm{Ker}(d^0) = M$.

<u>Pf. of Cor.</u> Set $I^0 = I_M$ from the theorem.

$M \hookrightarrow I^0 \to I^0/M =: M_1$. Apply the theorem to M_1, and get $I^1 = I_{M_1}$. $d^0 : I^0 \to I^1$ is defined by

$$\begin{tikzcd}
I^0 \arrow[rr, "d^0\ (\text{defn!})"] \arrow[dr, two heads] & & I^1 \arrow[rr, "d^1"] \arrow[dr, two heads] & & I^2 = I_{M_2}\ (\text{Thm}) \\
& I^0/M = M_1 \arrow[ur, hook] & & I^1/M_1 = M_2 \arrow[ur, hook] &
\end{tikzcd}$$

and continue. $\square$

(18.2) Proof of Thm 18.1. –

Step 1. – An R module, E, is said to be a <u>cogenerator</u> if $\forall\ M \in R\text{-mod}$ and $\underset{m\neq 0}{m \in M}$, $\exists\ \underset{(R\text{-linear})}{\varphi : M \to E}$ s.t. $\varphi(m) \neq 0$.

Lemma. – $\mathbb{Q}/\mathbb{Z}$ is a cogenerator in <u>Ab</u>. ($= \mathbb{Z}$-mod).

Proof. – Let A be an abelian group and $a \in A$, $a \neq 0$.

Consider the subgroup generated by a, say $A_1 \hookrightarrow A$.

Depending on whether a is of finite order or not we have two cases. – (i) $\underset{a \leftrightarrow 1}{A_1 \simeq \mathbb{Z}}$ (ii) $\underset{a \leftrightarrow 1}{A_1 \simeq \mathbb{Z}/N\mathbb{Z}}$

In any case there is a non-zero map (sending a to a non-zero element $\frac{1}{N} \in \mathbb{Q}/\mathbb{Z}$)

$$\begin{array}{ccccc} A_1 & \longrightarrow & \mathbb{Z}/N\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ a & \longmapsto & 1 & \longmapsto & \frac{1}{N} \end{array}$$

As $\mathbb{Q}/\mathbb{Z}$ is injective, we can lift this to
$$\begin{tikzcd} A_1 \arrow[r, hook] \arrow[d] & A \arrow[dl, "\exists"] \\ \mathbb{Q}/\mathbb{Z} \end{tikzcd}$$
and we are done. $\square$

Step 2. If $Q \in$ <u>Ab</u> is an injective cogenerator, then so is $I_Q = \mathrm{Hom}_{\mathbb{Z}}(R, Q) \in R$-mod.

Proof. - Given $M \in R\text{-mod}$ and $m \in M$ $(m \neq 0)$

we need to find $f \in \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbb{Z}}(R, Q))$ s.t. $f(m) \neq 0$

$$\begin{array}{c} \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbb{Z}}(R, Q)) \\ \big\downarrow \cong \quad \text{(Homework)} \\ \operatorname{Hom}_{\mathbb{Z}}(M, Q) \end{array}$$

Pick $\rho \in \operatorname{Hom}_{\mathbb{Z}}(M, Q)$ s.t. $\rho(m) \neq 0$. Let f be its preimage under

i.e. $f(x)(r) = \rho(r.x)$. Then $f(m) : R \longrightarrow Q$ $\mathbb{Z}$-linear

is a non-zero map since $f(m)(1) = \rho(m) \neq 0$.

Step 3. Direct product of injective modules is injective. [Homework].

Step 4. Let $I \in R\text{-mod}$ be an injective cogenerator. (exists by Steps 1 & 2)

For any $M \in R\text{-mod}$ define $I_M = \prod_{\varphi \in \operatorname{Hom}_R(M, I)} I_{(\varphi)}$ $\left(I_{(\varphi)} = I \ \forall \varphi\right)$

with
$$\begin{array}{ccc} i : M & \longrightarrow & I_M \\ \Psi & & \Psi \\ x & \longmapsto & (\varphi(x))_{\varphi \in \operatorname{Hom}_R(M, I)} \end{array}$$

This is clearly an R-linear map. It is injective because $\forall\, x \in M$, $x \neq 0$,

there exists $\psi : M \longrightarrow I$ s.t. $\psi(x) \neq 0$; i.e. ψ^{th} component of $i(x)$ is non-zero., hence $i(x) \neq 0$.

$\square$

(18.3) Projective / Free resolutions. –

Lemma. – For every set I, the direct sum $R^{(I)}$ $\left(= \bigoplus_{i \in I} R_{(i)}\right.$ each $R_{(i)} = R$ as an R-module$\left.\right)$ is projective.

(such R-modules are called <u>free</u>).

Proof. We need to prove that $\mathrm{Hom}_R(R^{(I)}, -)$ is exact. By defn. (of direct sum) $\mathrm{Hom}_R(R^{(I)}, X) = \prod_{i \in I} \mathrm{Hom}_R(R, X)$

and

$$\begin{array}{ccc} \mathrm{Hom}_R(R, X) & \simeq & X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \eta & \longmapsto & \eta(1) \end{array}$$

. Thus, we need to prove that for every exact seq. $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ of R-modules and a set I, the sequence $0 \to X^I \xrightarrow{f^I} Y^I \xrightarrow{g^I} Z^I \to 0$ is exact

(notation : $M^I = \prod_{i \in I} M$). This is an easy exercise. (Problem 7 HW4). $\square$

(18.4) **Proposition.** – $\forall\, M \in R\text{-mod}$, there exists a free object F and a surjective R-linear map $F \xrightarrow{\pi} M$.

Proof. Take $F = R^{(M)} \xrightarrow{\pi} M$ induced from

$$\begin{array}{ccccc} R & = & R & \longrightarrow & M \\ \uparrow & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ m^{th} & & 1 & \longmapsto & m \\ \text{copy} & & & & \end{array}$$

$\square$

(It will suffice to take any generating set in M)

Cor. Let $P \in R\text{-mod}$ be a projective R-module. Then there exists $P' \in R\text{-mod}$ s.t. $P \oplus P'$ is free.

Pf.– Use the proposition to get a surjective map $\underset{\text{free}}{F} \twoheadrightarrow P$; and then use Thm (12.2) page 2 to see that $F = P \oplus P'$. $\square$

(18.5) Just as the proof of the corollary of Thm 18.1, we get that every $M \in R\text{-mod}$, admits a (free) resolution:
(projective)

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[rr, "d_2"] \arrow[dr, two heads] & & F_1 \arrow[rr, "d_1"] \arrow[dr, two heads] & & F_0 \arrow[r, "\pi"] & M \arrow[r] & 0 \\
\cdots & & \mathrm{Ker}(d^0) \arrow[ur, hook] & & \mathrm{Ker}(\pi) \arrow[ur, hook] & & &
\end{tikzcd}$$

"Algorithmic construction": (1) Pick a set of generators of M, say X_0 and set $F_0 = R^{(X_0)} \xrightarrow{\pi} M$.

$$1 \in R \text{ at } x^{th} \text{ spot} \longmapsto x$$

(2) Work out generators of the $\mathrm{Ker}(\pi)$, say X_1 ; $F_1 := R^{(X_1)} \twoheadrightarrow \mathrm{Ker}(\pi)$ and continue. (syzygies!)

e.g. $R = K[x, y, z]$ (polynomial ring in 3 var) $\quad M = K$ with x, y, z acting as zero maps.

$$
\begin{tikzcd}
F_1 = R^{\oplus 3} \arrow[rr, "d_1"] \arrow[dr, "\pi_1"'] & & R = F_0 \arrow[r, "\pi"] & M = K \arrow[r] & 0 \\
& \mathrm{Ker}(\pi) = (x, y, z) \arrow[ur, hook] & 1 \arrow[r, mapsto] & 1 &
\end{tikzcd}
$$

$$F_1 = R\varepsilon_1 \oplus R\varepsilon_2 \oplus R\varepsilon_3 \xrightarrow{\pi_1} \mathrm{Ker}(\pi) \qquad \begin{cases} \varepsilon_1 \longmapsto x \\ \varepsilon_2 \longmapsto y \\ \varepsilon_3 \longmapsto z \end{cases}$$

Step 2. $\mathrm{Ker}(d_1) = \mathrm{Ker}(\pi_1)$ contains

$$y\varepsilon_1 - x\varepsilon_2$$
$$z\varepsilon_1 - x\varepsilon_3$$
$$\cancel{z/\varepsilon_2} \quad z\varepsilon_2 - y\varepsilon_3$$

$$F_2 = R\eta_1 \oplus R\eta_2 \oplus R\eta_3 \xrightarrow{\pi_2} \mathrm{Ker}(\pi_1)$$

$$\eta_1 \longmapsto y\varepsilon_1 - x\varepsilon_2$$
$$\eta_2 \longmapsto z\varepsilon_1 - x\varepsilon_3$$
$$\eta_3 \longmapsto z\varepsilon_2 - y\varepsilon_3$$

Step 3. $\mathrm{Ker}(\pi_2) = z\eta_1 - y\eta_2 + x\eta_3 \quad \left(\longmapsto \begin{array}{l} yz\cancel{\varepsilon_1} - xz\cancel{\varepsilon_2} \\ -y\cancel{z}\varepsilon_1 + xy\cancel{\varepsilon_3} = 0 \\ + xz\cancel{\varepsilon_2} - x\cancel{y\varepsilon_3} \end{array} \right)$

Thus we get

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} z \\ -y \\ x \end{bmatrix}} R^{\oplus 3} \xrightarrow{\begin{bmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{bmatrix}} R^{\oplus 3} \xrightarrow[{[x\ y\ z]}]{} R \longrightarrow 0$$

Checking exactness is a computation in linear algebra (use elimination to compute the kernel of a matrix.)