(19.0) Recall that we proved the following results for the abelian category $\mathcal{A} = R\text{-mod}$ (where as usual R is a commutative ring with $1 \neq 0$).

- Injective and projective resolutions exist.
- They are unique up to homotopy.

(19.1) Definition of Ext. – Injective resolutions.

Let $M, N \in \mathcal{A}$. Fix an injective resolution of N, say $I_N^\bullet$, and define, $\forall\, k \geq 0$

$$R^k h^M : \mathcal{A} \longrightarrow \mathcal{A} \quad \text{as}$$
$$N \longmapsto H^k\left(\mathrm{Hom}_{\mathcal{A}}(M, I_N^\bullet)\right) \quad \text{(on objects)}$$

for $N \xrightarrow{f} N'$, an R linear map, let

$$f^\bullet : I_N^\bullet \longrightarrow I_{N'}^\bullet$$ be a morphism of cochain complexes extending f (see Theorem 16.3 (page 3))

We get a morphism $\mathrm{Hom}_{\mathcal{A}}(M, I_N^\bullet) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M, I_{N'}^\bullet)$ of cochain complexes, inducing

$$R^k h^M(f) : H^k(\mathrm{Hom}(M, I_N^\bullet)) \longrightarrow H^k(\mathrm{Hom}(M, I_{N'}^\bullet))$$

Lemma. – The definition of $R^k h^M(f)$ does not depend on the choice of $f^\bullet$.

Proof. Let us first spell out the definition of $R^k h^M$ on objects. For any $N \in \mathcal{A}$, we pick an injective res.

$$0 \to (N \longrightarrow) I_N^0 \longrightarrow I_N^1 \to \cdots$$

Apply $\mathrm{Hom}(M, -)$ (also denoted by h^M - covariant hom functor)

$$0 \to \mathrm{Hom}(M, I_N^0) \to \mathrm{Hom}(M, I_N^1) \to \cdots$$

and set $$R^k h^M(N) := \frac{\mathrm{Ker}(\mathrm{Hom}(M, I_N^k) \to \mathrm{Hom}(M, I_N^{k+1}))}{\mathrm{Im}(\mathrm{Hom}(M, I_N^{k-1}) \to \mathrm{Hom}(M, I_N^k))}$$

To prove the lemma, let us first assume that

$$f = 0 : N \longrightarrow N'.$$

By Theorem 16.3, $f^\bullet$ is null homotopic. That is,

$$\exists \ s : I_N^k \longrightarrow I_{N'}^{k-1} \quad \text{s.t.} \quad ds + sd = f^\bullet$$

$\Rightarrow$ (as h^M is an additive functor) that

$h^M(s)$ is a homotopy between $h^M(\cancel{\mathrm{id}_{I_N}} f^\bullet)$ and zero morphism. By Prop. 15.4 (iii) page 7; we obtain

$$H^k\left(h^M(f^\bullet)\right) = R^k h^M(f) = 0.$$

The lemma now follows from the additivity of $\mathrm{Hom}(M,-)$ and H^k functors (see page 2 of Lecture 15). $\square$

Cor. The defn. of $R^k h^M(N)$ does not depend on the choice of the injective resolution $I_N^\bullet$ of N.

(Hint: Take $N' = N$ and $f = \mathrm{Id}_N$.)

(19.2) More conceptually, $\{R^k h^M : \mathcal{A} \longrightarrow \mathcal{A}\}_{k \geq 0}$ are obtained as composition:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\text{``}\mathcal{I}\text{''}"] & \mathbb{K}^\bullet(\mathcal{A}) \arrow[r, "h^M"] & \mathbb{K}^\bullet(\mathcal{A}) \arrow[r, "{\text{cohomology functors } \{H^k\}}"] & \mathcal{A} \\
N \arrow[r, mapsto] & I_N^\bullet \arrow[r, mapsto] & h^M(I_N^\bullet) \arrow[r] & \mathcal{A}
\end{tikzcd}$$

Caveat: The first arrow "$\mathcal{I}$" is NOT a functor. Choosing an injective resolution and mapping $N \longmapsto I_N^\bullet$ is allowed. However, there is an ambiguity in where to send morphisms. (the extn of $f \mapsto f^{\bullet}$ is only unique up to homotopy). The whole point of Prop 15.4 was to say that, even though there is an ambiguity at the 1^{st} step, it disappears when $H^k(\cdot)$ is tagged to it.

Alternately, one can define: $\overline{\mathbb{K}^{\bullet}(\mathcal{A})}$ to be the category $\mathbb{K}^{\bullet}(\mathcal{A})$ modulo null homotopic maps, i.e.,

Objects of $\overline{\mathbb{K}^{\bullet}(\mathcal{A})}$ = Objects of $\mathbb{K}^{\bullet}(\mathcal{A})$

$$\mathrm{Hom}_{\overline{\mathbb{K}^{\bullet}(\mathcal{A})}}(C^{\bullet}, D^{\bullet}) = \frac{\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(C^{\bullet}, D^{\bullet})}{\text{Null homotopic morphisms}}$$

Then Prop. 15.4 $\Rightarrow$ (i) $\overline{\mathbb{K}^{\bullet}(\mathcal{A})}$ is a category (i.e. composition of chain maps factors through null homotopic ones).

(ii) $N \longmapsto I_N^{\bullet}$ is a functor $\mathcal{A} \xrightarrow{\mathcal{I}} \overline{\mathbb{K}^{\bullet}(\mathcal{A})}$

and $R^k h^M = H^k \circ h^M \circ \mathcal{I}$ composition of additive functors, hence additive!

(19.3) Definition of Ext. — Projective Resolutions

$R^k h_N(M)$ is defined as: (i) Fix a projective res. $P_\bullet^M$ of M.

(so $\cdots \to P_2 \longrightarrow P_1 \longrightarrow P_0 \ (\to M) \to 0$) : chain complex.

(ii) Apply $h_N = \mathrm{Hom}(-, N)$

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \cdots$$ cochain complex

(iii) $$R^k h_N(M) := H^k(\mathrm{Hom}(P_\bullet, N))$$

— same arguments as in (19.2) apply, and we have additive functors $\{R^k h_N : \mathcal{A} \longrightarrow \mathcal{A}\}_{k \geq 0}$.

Before summarising the properties of these functors, we will prove the coincidence of two constructions.

(19.4) Theorem. — $\forall\, k \geq 0$; $M, N \in \mathcal{A}$, we have (natural) isomorphisms

$$R^k h^M(N) \simeq R^k h_N(M) \quad \left(=: \mathrm{Ext}^k_R(M,N)\right)$$

the ring

For $k = 0$, $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$

Proof. The theorem states that both

$$\begin{array}{ccc} \mathcal{A}\times\mathcal{A} & \longrightarrow & \mathcal{A} \\ (M,N) & \longmapsto & R^k h^M(N) \\ & \longmapsto & R^k h_N(M) \end{array}$$

are derived functors of the "bi functor" $\operatorname{Hom}_{\mathcal{A}}(-,-)$. Let us begin by proving this claim for $k=0$.

$\underline{R^0 h^M = h^M}$. $\forall\, N \in \mathcal{A}$, let $I_N^\bullet$ be an injective resolution

$$0 \to N \to I_N^0 \xrightarrow{d^0} \operatorname{Im}(d^0) \to 0$$ is exact and $h^M = \operatorname{Hom}(M,-)$ is left exact. So we get an exact sequence

$$0 \to \operatorname{Hom}(M,N) \longrightarrow \operatorname{Hom}(M, I_N^0) \xrightarrow{d^0\circ -} \operatorname{Hom}(M, \operatorname{Im}(d^0))$$

$$\Rightarrow \operatorname{Hom}(M,N) = \operatorname{Ker}(d^0\circ -) = R^0 h^M(N) \qquad \square$$

The proof of $R^0 h_N(M) = \operatorname{Hom}(M,N)$ is similar and hence omitted.

Strategy of proof. - We will produce another cochain complex $(C^\bullet, D)$ and show that $R^k h^M(N) \xrightarrow{\sim} H^k(C^\bullet) \xleftarrow{\sim} R^k h_N(M)$

So let $P_\bullet$ be a projective resolution of M and $I^\bullet$ be an injective resolution of N.

Define

$$\begin{tikzcd}
K^{p,q} := \mathrm{Hom}(P_p, I^q) \arrow[r, "d := -\circ d^P_{p+1}"'] \arrow[d, "\delta := d_I^q \circ -"] & K^{p+1,q} = \mathrm{Hom}(P_{p+1}, I^q) \\
K^{p,q+1} = \mathrm{Hom}(P_p, I^{q+1}) &
\end{tikzcd}$$

"vertical differential" δ ; "horizontal differential" d

(I am omitting superscripts from δ and d for convenience.)

Note: associativity of composition $\Rightarrow$ $d\delta = \delta d$. We can turn this equation into a differential:

$$C^n = \bigoplus_{\substack{p+q=n \\ p,q\geq 0}} K^{p,q} \xrightarrow{D} C^{n+1} \quad \text{defined as:}$$

$$\boxed{D(x) = d(x) + (-1)^p \delta(x)}$$

$$\forall\, x \in K^{p,q} \subset C^n \; ; \quad D(x) = \cancel{\delta}^{\,d}(x) + (-1)^p \cancel{d}^{\,\delta}(x) \in K^{p+1,q} \oplus K^{p,q+1} \subset C^{n+1}$$

$$\Big[\ \text{Check: } D(D(x)) \in K^{p+2,q} \oplus K^{p+1,q+1} \oplus K^{p,q+2}$$

$$\parallel$$

$$\cancel{\delta}_d(\cancel{\delta}_d(x)) + (-1)^{p+1} d(\delta(x)) + (-\cancel{d}^{\,\delta}(\cancel{d}^{\,\delta}(x))) + (-1)^{p+1}\delta(d(x))$$

$$= 0 \ \Big]$$

Claim: The canonical projection and inclusion

$$\begin{array}{c} C^n \underset{\hookleftarrow}{\twoheadrightarrow} K^{n,0} \\ \cup \\ \mathrm{Ker}(D^n) \end{array}$$

induces isomorphisms $\dfrac{\mathrm{Ker}(D^n)}{\mathrm{Im}(D^{n-1})} \simeq \dfrac{\mathrm{Ker}(\mathrm{Hom}(P_n, N) \to \mathrm{Hom}(P_{n+1}, N))}{\mathrm{Im}(\mathrm{Hom}(P_{n-1}, N) \to \mathrm{Hom}(P_n, N))}$

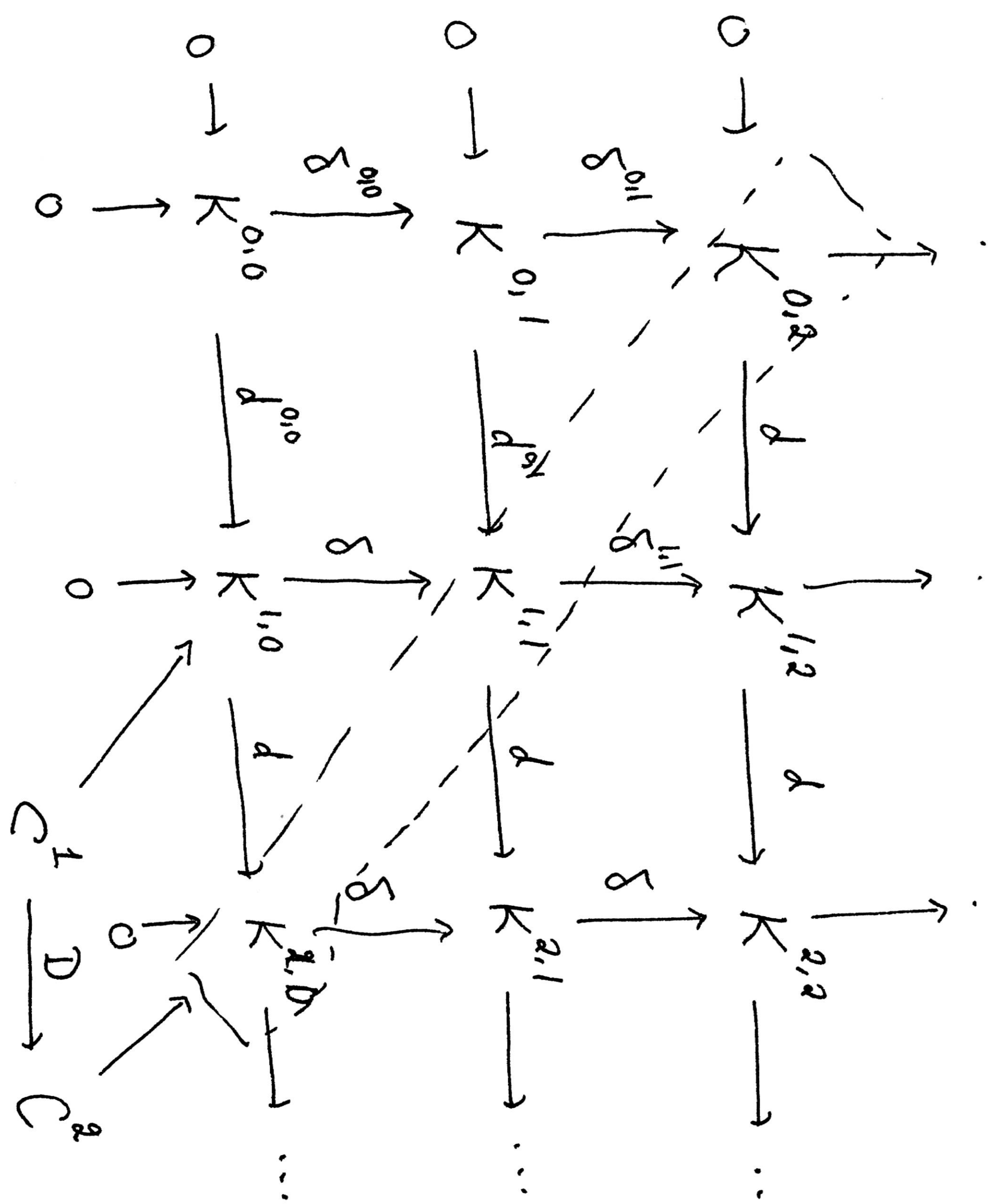

Figure (19.5) bicomplex $\{K^{p,q}\}_{p,q\geq 0}$ and associated single complex $\{C^n\}_{n\geq 0}$

(19.6) Some comments about Figure 19.5 page 8. –

(1) For fixed $p \geq 0$, the vertical differential $\{\delta^{p,q} : K^{p,q} \to K^{p,q+1}\}_{q \geq 0}$ forms the complex $\mathrm{Hom}(P_p, I^\bullet)$. As P_p is projective and $\mathrm{Hom}(P_p, -)$ is exact we get $\{\delta^{p,q}\}_{q\geq 0}$ is exact everywhere except $\mathrm{Ker}(\delta^{p,0}) = \mathrm{Hom}(P_p, N)$

(2) Similarly $\{d^{p,q}\}_{p\geq 0}$ form the complex $\mathrm{Hom}(P_\bullet, I^q)$ hence exact everywhere except $\mathrm{Ker}(d^{0,q}) = \mathrm{Hom}(M, I^q)$.

(3) $\mathrm{Ker}\left(\mathrm{Hom}(P_p, N) \to \mathrm{Hom}(P_{p+1}, N)\right) = \mathrm{Ker}(\delta^{p,0}) \cap \mathrm{Ker}(d^{p,0})$

(Similarly for the maps on the y-axis).

(19.7) Let us explicitly write an element of $C^n_{n,0}$ as

$$x = (x_0, x_1, \ldots, x_n) \in K^{0,n} \oplus \cdots \oplus K^{n,0}$$

$$D(x) = 0 \iff \delta(x_0) = 0,\ d(x_0) = \delta(x_1),\ \ldots,\ d(x_{n-1}) = \delta(x_n);\ d(x_n) = 0$$

We will show, by induction on r, that $(0 \leq r \leq n)$

$$x \equiv (0, \ldots, 0, \bar{x}_r, \bar{x}_{r+1}, \ldots, \bar{x}_n) \mod \mathrm{Im}(D^{n-1})$$

Base case: $r = 0$ obvious (tautology). Induction step: $(0, \ldots, 0, x_r, \ldots, x_n) \in \mathrm{Ker}(D^n)$

$\Rightarrow \delta(x_r) = 0;\ d(x_r) = \delta(x_{r+1});\ \ldots\ d(x_n) = 0$

As $\delta^{r,n-r}$ is exact for $1 \leq r \leq n-1$

$\exists\ y \in K^{r,n-r-1}$ s.t. $\delta(y) = x_r$ — see picture below.

Thus, mod $\operatorname{Im}(D^{n-1})$

$$\left(\begin{aligned} D(y) = \ & \delta(y) \in K^{r,n-r} \\ & + (-1)^r d(y) \in K^{r+1,n-r-1} \end{aligned} \right)$$

$$x \equiv \left(0, \dots, 0, x_{r+1} - (-1)^r d(y), x_{r+2}, \dots, x_n\right)$$

$$\begin{array}{llll}
(0,n) \ni 0, & & & \\
(1,n-1) \ni 0 \ \cdots & 0, & & \\
\ddots & (r,n-r) \ni x_r & & \\
& y \bullet \nearrow & (r+1,n-r-1) \ni x_{r+1} & \\
& & \ddots & \ddots \\
& & & (n,0) \ni x_n
\end{array}$$

Now $\quad x \equiv (\underbrace{0, \dots, 0}_{n}, \bar{x}_n) \longmapsto \bar{x}_n \in \operatorname{Ker}(\delta^{n,0}) \cap \operatorname{Ker}(d^{n,0})$

mod $\operatorname{Im}(D^{n-1})$

$$\operatorname{Ker}(\delta^{n,0}) \cap \operatorname{Ker}(d^{n,0}) = \operatorname{Ker}\left(\operatorname{Hom}(P_n, N) \to \operatorname{Hom}(P_{n+1}, N)\right)$$

Exercise. — We get an iso

$$\frac{\operatorname{Ker}(D^n)}{\operatorname{Im}(D^{n-1})} \simeq \frac{\operatorname{Ker}\left(\operatorname{Hom}(P_n,N) \to \operatorname{Hom}(P_{n+1},N)\right)}{\operatorname{Im}\left(\operatorname{Hom}(P_{n-1},N) \to \operatorname{Hom}(P_n,N)\right)} = R^n h_N(M)$$

Hint: the inverse map is given by

$$\lambda \in \operatorname{Ker}\left(\operatorname{Hom}(P_n,N) \to \operatorname{Hom}(P_{n+1},N)\right) \longmapsto (\overbrace{0, \dots, 0}^{n \text{ term}}, \lambda) \in \operatorname{Ker}(D^n)$$

$\square$