Lecture 20

(20.0) Summary of results so far. – Let R be a commutative ring with $1 \neq 0$; and let $\mathcal{A} = R$-mod. For $M, N \in \mathcal{A}$, we defined $\mathrm{Ext}^k_R(M,N) \in \mathcal{A}$ as follows.

(a) For any projective resolution of M; $\cdots \to P_2 \to P_1 \to P_0 \to 0$

$$\mathrm{Ext}^k(M,N) = \frac{\mathrm{Ker}(\mathrm{Hom}(P_k,N) \longrightarrow \mathrm{Hom}(P_{k+1},N))}{\mathrm{Im}(\mathrm{Hom}(P_{k-1},N) \to \mathrm{Hom}(P_k,N))}$$

or (b) For any injective resolution of N; $0 \to I^0 \to I^1 \to \cdots$

$$\mathrm{Ext}^k(M,N) = \frac{\mathrm{Ker}(\mathrm{Hom}(M,I^k) \longrightarrow \mathrm{Hom}(M,I^{k+1}))}{\mathrm{Im}(\mathrm{Hom}(M,I^{k-1}) \longrightarrow \mathrm{Hom}(M,I^k))}$$

The following theorem combines the results proved in last six lectures.

Theorem. (Part I) For a fixed $N \in R$-mod, we have (additive, contra functors $\mathrm{Ext}^k(-,N) : R\text{-mod} \longrightarrow R\text{-mod}$ $(\forall\, k \geq 0)$

such that

(i) $\mathrm{Ext}^0(-,N) \simeq \mathrm{Hom}(-,N)$

(ii) $M \in R$-mod is projective $\Rightarrow \mathrm{Ext}^k(M,N) = 0 \ \forall k \geq 1$

(iii) $\forall\ 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ short exact sequence in R-mod, we get a long exact sequence (again of R modules)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(M_3,N) \arrow[r] & \mathrm{Hom}(M_2,N) \arrow[r] & \mathrm{Hom}(M_1,N) \arrow[dll, out=0, in=180] \\
& \mathrm{Ext}^1(M_3,N) \arrow[r] & \mathrm{Ext}^1(M_2,N) \arrow[r] & \mathrm{Ext}^1(M_1,N) \arrow[dll, out=0, in=180] \\
& \cdots & & \cdots \\
\cdots \arrow[r] & \mathrm{Ext}^k(M_3,N) \arrow[r] & \mathrm{Ext}^k(M_2,N) \arrow[r] & \mathrm{Ext}^k(M_1,N) \arrow[dll, out=0, in=180] \\
& \mathrm{Ext}^{k+1}(M_3,N) \arrow[r] & \cdots
\end{tikzcd}$$

(iv) For every commutative diagram with exact rows, of R modules :

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1' \arrow[r] & M_2' \arrow[r] & M_3' \arrow[r] & 0
\end{tikzcd}$$

the following diagram commutes ($\forall\, k \geq 0$)

$$\begin{tikzcd}
\mathrm{Ext}^k(M_1,N) \arrow[r] & \mathrm{Ext}^{k+1}(M_3,N) \\
\mathrm{Ext}^k(M_1',N) \arrow[r] \arrow[u] & \mathrm{Ext}^{k+1}(M_3',N) \arrow[u]
\end{tikzcd}$$

Exercise. — State Theorem (Part II) for the sequence of functors $\mathrm{Ext}^k(M,-) : R\text{-mod} \longrightarrow R\text{-mod}$.

(20.1) Hints for the proof of Theorem 20.0 part I.

(i) Lecture 19 page 6.

(ii) Problem 3 of Homework # 6.

(iii) & (iv). Apply Theorem 14.3, page 3, to the following exact sequence of cochain complexes; where $I^{\bullet}$ is an injective resolution of N

$$0 \to \mathrm{Hom}(M_3, I^{\bullet}) \to \mathrm{Hom}(M_2, I^{\bullet}) \to \mathrm{Hom}(M_1, I^{\bullet}) \to 0$$

Note. – $\forall\, k \geq 0$, the sequence

$$0 \to \mathrm{Hom}(M_3, I^k) \to \mathrm{Hom}(M_2, I^k) \to \mathrm{Hom}(M_1, I^k) \to 0$$

is exact since I^k is injective.

Note. – (Lecture 19) – (a) and (b) of § 20.0 above yield the same result.

(20.2) An example $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. using projective res.

$R = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$, $N = \mathbb{Z}/n\mathbb{Z}$

(i) $0 \to \mathbb{Z} \xrightarrow[m]{\substack{\text{mult.} = \mu_m \\ \text{by}}} \mathbb{Z} \longrightarrow 0$ is a projective/free resolution of $\mathbb{Z}/m\mathbb{Z}$.

Apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/n\mathbb{Z})$ to get

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \arrow[r, "- \circ \mu_m"'] \arrow[d, equal] & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r, "\text{mult. by } m"'] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \\
& x \arrow[r, mapsto] \arrow[u, phantom, "\in" rotate=90] & mx \arrow[u, phantom, "\in" rotate=90] &
\end{tikzcd}$$

(Not exact! just a complex)

So $\mathrm{Ext}^0(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \{x \in \mathbb{Z}/n\mathbb{Z} \mid mx \equiv 0 \bmod n\}$

$$\mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \frac{\mathbb{Z}/n\mathbb{Z}}{x \sim y \text{ if } m \mid x-y}$$

$$= \frac{\mathbb{Z}/n\mathbb{Z}}{\{\text{classes mod } n \text{ of } m, 2m, 3m, \cdots, (n-1)m, 0\}}$$

Exercise (see problem 9, HW#1 and problem 9 of HW#11 of Algebra I at https://people.math.osu.edu/gautam.42/F17/algebra.html) Verify that both these groups have order $\gcd(m,n)$.

(20.3) Interpretation of $\mathrm{Ext}^1_R(M, N)$ via short exact sequences.

Let $\mathcal{S}(M; N)$ be the set of all short exact sequences

$$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0 \quad \text{of } R\text{-modules}$$

up to iso.

$$\left(\begin{tikzcd}[row sep=small]
 & & X \arrow[dd, "\cong"] \arrow[dr] & & \\
0 \arrow[r] & N \arrow[ur] \arrow[dr] & & M \arrow[r] & 0 \\
 & & X' \arrow[ur] & &
\end{tikzcd}\right)$$

Proposition. – $\mathrm{Ext}^1_R(M,N) \simeq \mathcal{S}(M;N)$ (bijection).

Proof.– Let us begin by unfolding the definition of Ext^1. We will see that only the first step of the projective resolution of M is needed.

Let $\cdots \to P_1 \to P_0 \to 0$ be a projective resolution of M.

By defn. $\quad \mathrm{Ext}^1(M,N) = \dfrac{\mathrm{Ker}(\mathrm{Hom}(P_1,N) \to \mathrm{Hom}(P_2,N))}{\mathrm{Im}(\mathrm{Hom}(P_0,N) \to \mathrm{Hom}(P_1,N))}$

i.e. $\xi \in \mathrm{Ext}^1(M,N)$ represents ~~an element~~ a morphism $P_1 \xrightarrow{\xi} N$ s.t.

$$\begin{tikzcd} P_2 \arrow[r, "d_2"] \arrow[dr, "0"'] & P_1 \arrow[d, "\xi"] \\ & N \end{tikzcd}$$

Such a morphism factors through (a unique)

$$\begin{tikzcd} P_1/\mathrm{Im}(d_2) = P_1/\mathrm{Ker}(d_1) \arrow[d, "\bar{\xi}"] \\ N \end{tikzcd}$$

Let $K = P_1/_{\mathrm{Im}(d_2)} = P_1/_{\mathrm{Ker}(d_1)} \xhookrightarrow{d_1} P_0$. Thus we get

a short exact sequence

$$0 \longrightarrow K \xrightarrow{i} P_0 \xrightarrow{\pi} M \to 0 \qquad (*)$$

and we have an identification: $\mathrm{Ext}^1(M,N) = \dfrac{\mathrm{Hom}(K,N)}{\{f\circ i \mid f \in \mathrm{Hom}(P_0,N)\}}$

[Alternately: Given a short exact sequence $(*)$, we get a long exact seq

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(P_0,N) \to \mathrm{Hom}(K,N) \to \mathrm{Ext}^1(M,N) \to \cancel{\mathrm{Ext}^1(P_0,N)} \to 0$$

(P_0: projective)

$$\Rightarrow \mathrm{Ext}^1(M,N) \simeq \frac{\mathrm{Hom}(K,N)}{\mathrm{Im}(-\circ i)} \quad]$$

The goal of the proof of the proposition of previous page is to obtain an element $\overline{\xi}$ of $\dfrac{\mathrm{Hom}(K,N)}{\mathrm{Im}(-\circ i)}$, given a short exact sequence

$$\xi : \; 0 \to N \xrightarrow{f} X \xrightarrow{g} M \to 0 \quad ; \quad \text{and vice verca.}$$

Given a short exact sequence ξ as above, consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, dashed, "\overline{\xi}"] & P \arrow[r, "\pi"] \arrow[d, dashed, "\psi"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "f"'] & X \arrow[r, "g"'] & M \arrow[r] & 0 \end{tikzcd}$$

- $\psi : P \to X$ exists because P is projective:

$$\begin{tikzcd} & P \arrow[d, "\pi"] \arrow[dl, dashed] & \\ X \arrow[r, "g"'] & M \arrow[r] & 0 \end{tikzcd}$$

- $g\psi i = \mathrm{id}_M \pi i = 0 \Rightarrow \psi i$ factors through $f\overline{\xi}$. ($\overline{\xi}$ is unique such)

The assignment $\xi \mapsto \overline{\xi}$ is only defined up to the choice of ψ. We claim that the effect of changing ψ to ψ_1 (& hence $\overline{\xi}$ to $\overline{\xi}_1$) is immaterial modulo $\mathrm{Im}(-\circ i)$. This is because $g(\psi - \psi_1) = 0 \Rightarrow \psi - \psi_1$ factors through f (ie $\psi - \psi_1 = \cancel{kf}$ fk for some $k : P \to N$)

and hence

$$(\psi - \psi_1)\circ i = fki \; (=) \; f\circ(\overline{\xi} - \overline{\xi}_1)$$

As f is injective $\overline{\xi} - \overline{\xi}_1 = ki \in \mathrm{Im}(-\circ i)$.

In order to show that $\xi \longmapsto \overline{\xi}$, we will have to verify that

$$\xi \cong \eta \implies \overline{\xi} = \overline{\eta}.$$

(two short exact sequences)

This follows from the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & N \arrow[rr] \arrow[dd, equal] & & X \arrow[rr] \arrow[dd, "\alpha", "\cong"'] & & M \arrow[r] \arrow[dr, equal] & 0 & & : \xi \\
& 0 \arrow[r] & K \arrow[ul, "\bar{\xi}"'] \arrow[dl, "\bar{\eta}"] \arrow[rr] & & P \arrow[ul, "\psi"'] \arrow[dl, "\alpha\psi"] \arrow[rr] & & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[rr] & & \tilde{X} \arrow[rr] & & M \arrow[r] \arrow[ur, equal] & 0 & & : \eta
\end{tikzcd}
$$

Thus we have a well-defined map

$$
\begin{array}{ccc}
S(M;N) & \longrightarrow & \dfrac{\mathrm{Hom}(K,N)}{\mathrm{Im}(-\circ i)} = \mathrm{Ext}^1(M,N) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\xi & \longmapsto & \bar{\xi}
\end{array}
$$

To prove this map is a bijection, we will write an explicit inverse

Let $h \in \mathrm{Hom}(K,N)$. We produce a short exact sequence as follows:

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "h"'] & P \arrow[r, "\pi"] \arrow[d, "\psi"] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r, "f"'] & X
\end{tikzcd}
$$

Set $X = \dfrac{N \oplus P}{\{(h(k), -i(k)) : k \in K\}}$

$f : N \to X$ defined by

$N \hookrightarrow N \oplus P \twoheadrightarrow X$

(similarly $P \xrightarrow{\psi} X$ s.t. $\psi \circ i = f \circ h$)

Ex. f is injective.

Now consider

$$
\begin{tikzcd}
N \oplus P \arrow[r] \arrow[d] & P \arrow[r, "\pi"] & M \\
X \arrow[urr, dashed]
\end{tikzcd}
$$

Check (exercise) that $X \to M$ is surjective with Kernel $= N \overset{f}{\hookrightarrow} X$

[More abstractly: X satisfies the universal property of a push-forward

diagram
$$\begin{tikzcd} K \arrow[r] \arrow[d] & P \arrow[d] \arrow[ddr, bend left] & \\ N \arrow[r] \arrow[drr, bend right] & X \arrow[dr, dashed, "\exists!"] & \\ & & Y \end{tikzcd}$$
. Take $Y = M$

$P \to M = \pi$

$N \to M =$ zero morphism

We have already done this exercise!]

$\square$

(20.4) Example $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

Projective resolution of $\mathbb{Z}/n\mathbb{Z}$: $0 \to \mathbb{Z} \xrightarrow[\text{mult. by } n]{\mu_n} \mathbb{Z} \to 0$

Complex: $0 \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{-\circ\mu_n} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0$

$$\begin{matrix} & & \| & & \| & & \\ 0 & \to & \mathbb{Z} & \xrightarrow[\mu_n]{\qquad} & \mathbb{Z} & \to & 0 \end{matrix}$$

$\mathrm{Ext}^0(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathrm{Ker}(\mu_n) = 0 \quad (= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}))$

$\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/\mathrm{Im}(\mu_n) = \mathbb{Z}/n\mathbb{Z}$.

Meaning: up to isomorphism, there are exactly n short exact seq

$$0 \to \mathbb{Z} \longrightarrow X \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$$

More explicitly, $\forall\, k \in \mathbb{Z}/n\mathbb{Z}$, we get $X_k = \dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\text{submod. gen. by } (k,n)}$

$\left\{ 0 \to \mathbb{Z} \longrightarrow X_k \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0 \right\}_{0 \leq k \leq n-1}$ – set of all such s.e.s's