Lecture 21

(21.0) Cohomology theories (e.g. de Rham cohomology, sheaf cohomology, Čech cohomology, Lie algebra cohomology, Hochschild cohomology ...) are often viewed as derived functors of something. Here we introduce group cohomology, for its potential applications to Galois theory.

Let G be a group (not necessarily abelian) and let M be an abelian group, together with a G-action (as usual I will write it as $G \circlearrowright M$). This means we are given a group hom.
"G acts on M"

$$G \xrightarrow{\tau} \mathrm{Aut}_{\mathbb{Z}}(M)$$

and one often omits τ from the notation, and write

$$g \cdot m = \tau(g)(m) \quad \forall\ g \in G,\ m \in M$$

(21.1) Define $C^n(G;M)$ = set of all maps (of sets)

$$f : \underbrace{G \times \cdots \times G}_{n\text{-factors}} \longrightarrow M$$

$C^n(G;M)$ has a structure of an abelian group, borrowed from M

$$\left(\text{i.e., } (f_1 + f_2)(g_1, \dots, g_n) := f_1(g_1, \dots, g_n) \underset{\text{in } M}{+} f_2(g_1, \dots, g_n)\right)$$

$$\begin{array}{ccc} d : C^n(G;M) & \longrightarrow & C^{n+1}(G;M) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & df \end{array} \quad \text{defined by}$$

$$\begin{aligned} (df)(g_0, \dots, g_n) = \ & g_0 \cdot f(g_1, \dots, g_n) \\ & + \sum_{i=0}^{n-1} (-1)^{i+1} f(g_0, \dots, g_{i-1}, \underbrace{g_i g_{i+1}}, g_{i+2}, \dots, g_n) \\ & + (-1)^{n+1} f(g_0, \dots, g_{n-1}) \end{aligned}$$

product of $g_i g_{i+1}$ in G

Check: $d(df) = 0$. Thus we get a cochain complex $\{C^n(G;M), d\}_{n \geq 0}$.

$$\boxed{\begin{aligned} H^n(G;M) &= \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1}) \\ &= n^{\text{th}} \text{ cohomology of } \{C^\bullet(G;M), d\} \\ &\text{Cohomology of } G \text{ with coefficients from } M \end{aligned}}$$

(21.2) Small values of n (0, 1 and 2).

$n = 0$.
$$0 \longrightarrow \underset{\substack{\| \\ M}}{C^0(G;M)} \xrightarrow{d} \underset{\substack{\| \\ \{G \to M\}}}{C^1(G;M)}$$

$$(d(m))(\sigma) = \sigma \cdot m - m$$

So $H^0(G;M) = \{m \in M \mid \sigma \cdot m = m \ \forall \sigma \in G\} = M^G$

$n=1$. An element of $H^1(G;M)$ is represented by a set map $f: G \to M$ s.t. $(df)(\sigma_1,\sigma_2)=0$. By defn. this means:

$$\sigma_1 \cdot f(\sigma_2) - f(\sigma_1\sigma_2) + f(\sigma_1) = 0.$$

i.e. $$f(\sigma_1\sigma_2) = \sigma_1 \cdot f(\sigma_2) + f(\sigma_1)$$

Recall the definition of the semidirect product $M \rtimes_\tau G$:

as a set it is $M \times G$; with the group operation

$$(m_1,\sigma_1)(m_2,\sigma_2) = (m_1 + \sigma_1 \cdot m_2, \sigma_1\sigma_2)$$

Claim: $$\begin{aligned} G &\longrightarrow M \rtimes_\tau G \\ \sigma &\longmapsto (f(\sigma),\sigma) \end{aligned}$$ is a group hom iff $df = 0$

Proof. –
$$\begin{aligned} \sigma_1\sigma_2 &\longmapsto (f(\sigma_1\sigma_2), \sigma_1\sigma_2) \\ &\longmapsto (f(\sigma_1),\sigma_1)\cdot(f(\sigma_2),\sigma_2) \\ &= (f(\sigma_1) + \sigma_1\cdot f(\sigma_2), \sigma_1\sigma_2) \end{aligned}$$

$\square$

So we have a bijection between

$$\underset{\substack{\| \\ Z^1(C^\cdot(G;M))}}{\{f: G \to M \mid df = 0\}} \longleftrightarrow \left\{ \begin{array}{l} \text{Group homs. } G \xrightarrow{\xi} M \rtimes G \\ \text{s.t. } \pi \circ \xi = \mathrm{Id}_G \text{ where} \\ M \rtimes G \xrightarrow{\pi} G \\ (m,\sigma) \longmapsto \sigma \end{array} \right\}$$

$\longleftrightarrow$ Sections 's' of the short exact sequence (of groups – not necessarily abelian)

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] & M \rtimes G \arrow[r, "\pi"] & G \arrow[r] \arrow[l, bend left, dashed, "s"] & \{1\} \end{tikzcd}$$

Claim 2. $f : G \to M$ is of the form $f(\sigma) = \sigma.m - m$ for some $m \in M$ iff

$$G \to M \rtimes G,\quad \sigma \mapsto (f(\sigma), \sigma)$$
$$G \to M \rtimes G,\quad \sigma \mapsto (0, \sigma)$$

are related by Conjugation with $(m, e) = i(m)$.

Pf. Let us write $G \xrightarrow{j} M \rtimes G$, $\sigma \longmapsto (0,\sigma)$. Conjugating by $i(m)^{-1}$ changes j to

$$\begin{aligned} \tilde{j}(\sigma) &= (m, e)^{-1}\,(0,\sigma)\,(m, e) \\ &= (-m, e)\,(\sigma.m, \sigma) \\ &= (\sigma.m - m, \sigma) = (f(\sigma), \sigma) \end{aligned}$$

$\Longleftrightarrow \quad f(\sigma) = \sigma.m - m$

$\square$

$$H^1(G; M) = \frac{\text{Sections } (s) \text{ of } 0 \to M \xrightarrow{i} M \rtimes G \rightleftarrows G \to \{1\}}{\text{Conj by } i(M)}$$

(21.3) $n=2$. An element of $Z^2(G;M)$ is a function

$f : G \times G \longrightarrow M$ satisfying

$$\sigma_1 \cdot f(\sigma_2,\sigma_3) - f(\sigma_1\sigma_2,\sigma_3) + f(\sigma_1,\sigma_2\sigma_3) - f(\sigma_1,\sigma_2) = 0$$

i.e. $$\boxed{\sigma_1 \cdot f(\sigma_2,\sigma_3) + f(\sigma_1,\sigma_2\sigma_3) = f(\sigma_1\sigma_2,\sigma_3) + f(\sigma_1,\sigma_2)} \quad - (*)$$

$$\forall\, \sigma_1,\sigma_2,\sigma_3 \in G$$

For $h : G \to M$, $dh \in B^2(G;M)$ is given by

$$(dh)(\sigma_1,\sigma_2) = \sigma_1 \cdot h(\sigma_2) - h(\sigma_1\sigma_2) + h(\sigma_1)$$

So let us assume we have $f_0 \in Z^2(G;M)$ s.t.

$$f_0(\sigma,e) = f_0(e,\sigma) = 0 \quad \forall\, \sigma \in G$$

Remark.— Modulo term of the form dh, we can always make sure this is the case. To see this, let $f \in Z^2(G;M)$

$\sigma_1 = \sigma_2 = e\,;\ \sigma_3 = \sigma$ in $(*)$ gives $\cancel{f(e,\sigma)}$ $f(e,\sigma) = f(e,e)\ \forall\, \sigma \in G$ $(=: x \text{ say})$

$\sigma_1 = \sigma\,;\ \sigma_2 = \sigma_3 = e$ in $(*)$ $\Rightarrow$ $f(\sigma,e) = \sigma . x$

Take any $h : G \to M$ s.t. $h(e) = x$. Then

$$(dh)(e,\sigma) = h(\sigma) - h(\sigma) + h(e) = x \quad \forall\, \sigma \in G$$

$$(dh)(\sigma,e) = \sigma . x - h(\sigma) + h(\sigma) = \sigma . x \quad \forall\, \sigma \in G$$

$\Rightarrow$ $f - dh =: f_0$ is s.t. $f(\sigma,e) = f(e,\sigma) = 0$.

Define $E_{f_0} = M \times G$ as a set; with

$$E_{f_0} \times E_{f_0} \longrightarrow E_{f_0} \quad \text{given by}$$

$$(m_1, \sigma_1) \underset{f_0}{\cdot} (m_2, \sigma_2) := \left(m_1 + \sigma_1(m_2) + f_0(\sigma_1, \sigma_2),\ \sigma_1\sigma_2\right)$$

Lemma. – E_{f_0} is a group with $(0, e)$ being the identity

Proof. – $(0, e)$ is neutral w.r.t. multiplication $\underset{f_0}{\cdot}$ because

$$f_0(e, \sigma) = f_0(\sigma, e) = 0 \quad \forall \sigma \in G.$$

Inverse : $$(m, \sigma)^{-1} = \left(-\sigma^{-1}(m) - f_0(\sigma^{-1}, \sigma),\ \sigma^{-1}\right)$$

Associativity $\Longleftrightarrow$ $df_0 = 0$ because

$$[(m_1\, \sigma_1)(m_2\, \sigma_2)](m_3\, \sigma_3) = \left(\begin{array}{l} m_1 + \sigma_1(m_2) + \sigma_1\sigma_2(m_3) \\ + f_0(\sigma_1, \sigma_2) + f_0(\sigma_1\sigma_2, \sigma_3) \end{array} ;\ \sigma_1\sigma_2\sigma_3\right)$$

$$(m_1\, \sigma_1)\,[(m_2, \sigma_2)(m_3, \sigma_3)] = \left(\begin{array}{l} m_1 + \sigma_1(m_2) + \sigma_1\sigma_2(m_3) \\ + \sigma_1 \cdot f_0(\sigma_2, \sigma_3) + f_0(\sigma_1, \sigma_2\sigma_3) \end{array} ;\ \sigma_1\sigma_2\sigma_3\right)$$

$\square$

Now, for $h_0 : G \longrightarrow M$ s.t. $h_0(e) = 0$, we can write

$$M \underset{\tau}{\rtimes} G \longleftarrow E_{f_0}$$

$$(m + h_0(\sigma), \sigma) \longleftarrow\!\shortmid\ (m, \sigma)$$

Check : We have an iso. of groups $\Longleftrightarrow$ $f_0 = dh_0$.

In conclusion, one can prove that $H^2(G;M)$ is iso. to the group of extensions $0 \to M \to E_{(f_0)} \to G \to \{1\}$

[with an assumption that conjugation by (a lift $(0,\sigma)$ of) an element $\sigma \in G$, is same as $\tau(\sigma)$ on M.]

(21.4) $H^\bullet(G;M)$ as Ext.

$\mathcal{A}$ = category of abelian groups with G-action. Consider

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathcal{Ab} \\ M &\longmapsto M^G := \{x \in M \mid \sigma . x = x \ \forall \sigma \in G\} \end{aligned}$$

$$M^G = \operatorname{Hom}_{\mathcal{A}}(\text{Trivial}, M)$$

Trivial: $\mathbb{Z}$ as an abelian group

$$\begin{aligned} G &\xrightarrow{\tau} \operatorname{Aut}_{\mathbb{Z}}(\mathbb{Z}) \\ \sigma &\longmapsto \operatorname{Id}_{\mathbb{Z}} \end{aligned}$$

So, $H^\bullet(G;M)$ can be viewed as $\operatorname{Ext}^\bullet_{\mathcal{A}}(\text{Trivial}, M)$.

The precise argument is beyond our scope, but boils down to the following being a projective resolution of Trivial.

P_n $(n \geq 0)$ = free abelian group over $\underbrace{G \times \dots \times G}_{n+1 \text{ terms}}$. That is, an element of P_n is of the form (unique) $\sum a_{g_1,\dots,g_{n+1}} \varepsilon_{g_1,\dots,g_{n+1}}$

$\uparrow$
$\mathbb{Z}$ almost all zero, i.e. the sum is finite.

$G \circlearrowright P_n$ by $\sigma . (\varepsilon_{g_0,\dots,g_n}) = \varepsilon_{\sigma g_0, g_1, \dots, g_n}$

Differential $d : P_n \longrightarrow P_{n-1}$

$$d(\varepsilon_{g_0,\dots,g_n}) = \sum_{i=0}^{n-1} (-1)^i \varepsilon_{g_0,\dots,g_{i-1},\boxed{g_i g_{i+1}},\dots,g_n}$$

$$+ (-1)^n \varepsilon_{g_0,\dots,g_{n-1}}$$

i.e. each $\{P_n\}_{n \geq 0}$ is projective in $\mathcal{A}$; and the following sequence is exact:

$$\begin{array}{ccccccccc} \dots \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & \text{Trivial} & \longrightarrow 0 \\ & & & & & \| & & \| & \\ & & & & & \mathbb{Z}G & \longrightarrow & \mathbb{Z} & \\ & & & & & \varepsilon_g & \longmapsto & \begin{cases} 1 & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases} & \end{array}$$

Moreover $\operatorname{Hom}_{\mathcal{A}}(P_n, M)$ = all set maps $\underbrace{G \times \dots \times G}_{n \text{ factors}} \longrightarrow M$

$\quad \cup\!\!\!\!|$

$\eta \longmapsto f(g_1,\dots,g_n) = \eta(\varepsilon_{e,g_1,\dots,g_n})$