Lecture 22

(22.0) Let R be a commutative ring with $1 \neq 0$. Recall that $\forall\, M, N \in R\text{-mod}$, we have $M \underset{R}{\otimes} N \in R\text{-mod}$.

Properties (from Lecture 13):

(i) Universal property. We have an R-bilinear map $i : M \times N \longrightarrow M \underset{R}{\otimes} N$, s.t. for every bilinear map $M \times N \xrightarrow{f} X$ $\exists !$ R-linear map $g : M \underset{R}{\otimes} N \longrightarrow X$, s.t.

$$\begin{tikzcd} M \times N \arrow[r, "i"] \arrow[d, "f"'] & M \otimes N \arrow[dl, dashed, "g"] \\ X & \end{tikzcd}$$

This sets up a bijection ($\forall\, X \in R\text{-mod}$)

$$\mathrm{Bilinear}_R(M \times N ; X) \xleftarrow{\text{bijection}} \mathrm{Hom}_R(M \underset{R}{\otimes} N, X)$$

$$g \circ i \longleftarrow\!\shortmid\; g$$

(ii) Explicitly, an element of $M \otimes N$ is a finite sum of the form $\displaystyle\sum_{i=1}^{p} m_i \otimes n_i$ where $m_i \in M$, $n_i \in N$ $(\forall\, 1 \leq i \leq p)$.

Rules of manipulation.- $\left(\underset{\substack{\uparrow \\ \text{of } M}}{0} \otimes n = \underset{\substack{\uparrow \\ \text{of } M \otimes N}}{0} = m \otimes \underset{\substack{\uparrow \\ \text{of } N}}{0} \right)$

Distributive over addition $\quad (m + m') \otimes n = m \otimes n + m' \otimes n$

$$m \otimes (n + n') = m \otimes n + m \otimes n'$$

Scalars: $\quad (rm) \otimes n = m \otimes (rn)$

$(\forall\, r \in R ;\ m, m' \in M ;\ n, n' \in N)$

(iii) $- \otimes_R N \; : \; R\text{-mod} \longrightarrow R\text{-mod}$ is an

$(= N \otimes_R -)$ additive, right exact functor.

(iv) $$\left(\bigoplus_{i \in I} M_i \right) \otimes_R N \simeq \bigoplus_{i \in I} (M_i \otimes_R N)$$

(v) ∀ short exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$

$M_1 \otimes N \longrightarrow M_2 \otimes N \longrightarrow M_3 \otimes N \longrightarrow 0$ is exact.

(22.1) Reason for the failure of left-exactness.

We have seen an example of an injective morphism $M_1 \hookrightarrow M_2$ of R-modules such that $M_1 \otimes N \longrightarrow M_2 \otimes N$ is not injective.

($(2\mathbb{Z} \hookrightarrow \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ gives $\mathbb{Z}/2\mathbb{Z} \xrightarrow[\substack{\| \\ \text{mult by} \\ 2}]{0} \mathbb{Z}/2\mathbb{Z}$).

So it may happen that for an ideal $\mathfrak{a} \subset R$,

$\mathfrak{a} \otimes_R N \longrightarrow \mathfrak{a} \cdot N$ is not injective

(it is clearly surjective)

For instance if $a \in \mathfrak{a}$, $n \in N$ are such that $a \cdot n = 0$

but a is not of the form $a' \cdot x$ for $a' \in \mathfrak{a}$, $x \in \text{Ann}(n)$

Be careful in viewing an ideal as a module over R !

(22.2) Definition. An R-module N is said to be *flat* if $- \otimes_R N$ is an exact functor.

Some examples. – (1) $\boxed{\text{Free} \Rightarrow \text{Flat}}$. This is because

- R is flat as an R-module (recall: $X \otimes_R R \simeq X \quad \forall X \in R\text{-mod}$)
- $\otimes$ distributes over $\oplus$
- $\oplus$ preserves exactness. (Problem 7, HW4)

(2) $\boxed{\text{Projective} \Rightarrow \text{Flat}}$

Since we have: projective $\Leftrightarrow$ a direct summand of free (Problem 1 HW 6)

and we can prove an easy lemma:

Lemma: $N = N_1 \oplus N_2$ is flat $\Leftrightarrow$ N_1 and N_2 are flat.

Pf. Let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ be a short exact sequence of R-modules. Then

$$\begin{tikzcd}
0 \arrow[r] & M_1 \otimes N \arrow[r, "f \otimes Id_N"] \arrow[d, equal] & M_2 \otimes N \arrow[r, "g \otimes Id_N"] \arrow[d, equal] & M_3 \otimes N \arrow[r] \arrow[d, equal] & 0 & -(\mathrm{I}) \\
0 \arrow[r] & \begin{matrix} M_1 \otimes N_1 \\ \oplus \\ M_1 \otimes N_2 \end{matrix} \arrow[r, "{\begin{bmatrix} f \otimes Id_{N_1} & 0 \\ 0 & f \otimes Id_{N_2} \end{bmatrix}}"'] & \begin{matrix} M_2 \otimes N_1 \\ \oplus \\ M_2 \otimes N_2 \end{matrix} \arrow[r, "{\begin{bmatrix} g \otimes Id_{N_1} & 0 \\ 0 & g \otimes Id_{N_2} \end{bmatrix}}"'] & \begin{matrix} M_3 \otimes N_1 \\ \oplus \\ M_3 \otimes N_2 \end{matrix} \arrow[r] & 0 & -(\mathrm{II})
\end{tikzcd}$$

Thus N is flat $\iff$ (I) is exact $\iff$ (II) is exact
$(\forall M_1, M_2, M_3)$

$\iff$ each $- \otimes N_1$ & $- \otimes N_2$ is exact

$\iff$ N_1 and N_2 are flat

$\square$

(3) $\mathbb{Q}$ is flat over $\mathbb{Z}$. (Proof after Theorem 22.5 below).
but not free.

(4) $\mathbb{Q}/\mathbb{Z}$ is not flat (but injective, and divisible).

$$\left[\left(\mathbb{Z} \xhookrightarrow[\mu_2]{} \mathbb{Z} \right) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} \xrightarrow{0} \mathbb{Q}/\mathbb{Z} \right].$$

(22.3) More generally, if R is an integral domain, ~~and~~ we have another notion of "torsion-free" R modules.

Definition. For an R-module N,

$$N_{tor} := \{ n \in N \text{ s.t. } \exists\, r \in R \setminus \{0\} \text{ s.t. } r \cdot n = 0 \}$$

We say N is a torsion-free module if $N_{tor} = 0$.

If $N_{tor} \neq (0)$, we say "N has torsion".

Lemma. (R: integral domain) N has torsion $\Rightarrow$ N is not flat.

Pf. Let $a \in R$ $(a \neq 0)$ be such that there is $n \neq 0$, $n \in N$ with $a \cdot n = 0$

$$0 \to R \xrightarrow{\mu_a} R \left(\longrightarrow R/(a) \to 0 \right) \quad \left[\operatorname{Ker}(\mu_a) = \{0\} \right.$$

because R does not have zero divisors]

Tensoring with N yields

still mult. by a.

$$N \xrightarrow{\mu_a} N \quad \text{and} \quad n \in \operatorname{Ker}(\mu_a \text{ on } N) \text{, so it is}$$

not injective.

□

Remark. - There exist torsion-free non-flat modules. A standard example is $\mathfrak{a} = (x,y) \subset k[[x,y]]$ (k : a field). Proof that $\mathfrak{a}$ is not flat will appear later.

(22.4) Flatness via localization.

Again let R be an arbitrary comm. ring and let $S \subset R \setminus \{0\}$ be a multiplicatively closed set (i.e. $s,t \in S \Rightarrow s.t \in S$).

Lemma. $N = S^{-1}R$ (as an R-module for now) is flat.

Proof. Exercise.- review localization (Lecture 30 of Algebra I)

- $M \underset{R}{\otimes} S^{-1}R = S^{-1}M$ (HW12, Problem 1 of Algebra I).
- $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ exact

$$\Rightarrow \quad 0 \to S^{-1}M_1 \longrightarrow S^{-1}M_2 \longrightarrow S^{-1}M_3 \to 0 \quad \text{is exact}$$

[Prop. 31.2 of Algebra I ; page 3]

□

(22.5) Theorem. — $N \in R\text{-mod}$ is flat if, and only if for every ideal $\mathfrak{a} \subset R$, the R-linear map $\mathfrak{a} \otimes N \to \mathfrak{a}N$ is an isomorphism.

Proof. ($\Rightarrow$) Consider $0 \to \mathfrak{a} \longrightarrow R \longrightarrow R/\mathfrak{a} \longrightarrow 0$

Tensor with N to get $0 \to \mathfrak{a} \otimes N \longrightarrow R \otimes N \longrightarrow R/\mathfrak{a} \otimes N \to 0$

Exercise.— $R/\mathfrak{a} \underset{R}{\otimes} N = N/\mathfrak{a}N$. ($\bar{r} \otimes n \longmapsto \overline{rn}$)

i.e. for any N, we would have a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a}.N \arrow[r] & N \arrow[r] & N/\mathfrak{a}N \arrow[r] & 0 & (\text{exact } \forall N) \\
0 \arrow[r] & \mathfrak{a} \otimes N \arrow[r] \arrow[u] & N \arrow[r] \arrow[u, equal] & N/\mathfrak{a}N \arrow[r] \arrow[u, equal] & 0 & \text{exact for flat } N
\end{tikzcd}$$

$\Rightarrow$ $\mathfrak{a} \otimes N \longrightarrow \mathfrak{a}.N$ is an iso.

($\Leftarrow$) Assume $\mathfrak{a} \otimes N \to \mathfrak{a}.N$ is an isomorphism $\forall$ ideal $\mathfrak{a} \subset R$.

We want to prove that N is flat. So, let $M_1 \hookrightarrow M_2$ be any injective R-linear map.

Let F be a free R-module (say $F = R^{(J)} = \bigoplus_{j \in J} R$ for some indexing set J) s.t. $F \overset{\pi}{\twoheadrightarrow} M_2$ and let $K = \operatorname{Ker}(\pi)$, i.e.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"'] & F \arrow[r, "\pi"'] & M_2 \arrow[r] & 0 \\
 & & & M_1 \arrow[u, hook] &
\end{tikzcd}$$

Thus we get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_1 \arrow[r] \arrow[d] & F_1 := \pi^{-1}(M_1) \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "i"'] \arrow[d] & F \arrow[r, "\pi"'] & M_2 \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

iso. $(K_1 = K)$ $\rightsquigarrow$

$\Bigg\} \quad \otimes N$

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \cancel{\otimes} 0 \arrow[r] \arrow[d, "ⓘ"] & \mathrm{Ker}(f \otimes 1) \arrow[d] & \\
 & K \otimes N \arrow[r] \arrow[d, equal] & F_1 \otimes N \arrow[r] \arrow[d] & M_1 \otimes N \arrow[r] \arrow[d, "f \otimes 1"] & 0 \\
0 \arrow[r, "?"] & K \otimes N \arrow[r] \arrow[d] & F \otimes N \arrow[r] & M_2 \otimes N & \\
 & 0 & & &
\end{tikzcd}$$

Claim. For a free R-module F and an injective map $K \hookrightarrow F$, the map $K \otimes N \longrightarrow F \otimes N$ is injective.

[Claim + Snake Lemma $\Rightarrow$ $\mathrm{Ker}(f \otimes 1) = 0$, because maps labelled by ⓘ would be injective].

Proof of ~~the claim. – First we reduce prove it to for F being of finite rank. An easy induction argument will prove the claim, assuming we can show it for two~~

The proof of the claim follows from Lemma (22.6) below.

The lemma, together with our hypothesis on N implies the claim for $F = R^{(J)}$ when $|J| < \infty$; by induction. In general, if $K \hookrightarrow F = R^{(J)}$ (J not necessarily finite), and if $\sum_{i=1}^{p} k_i \otimes n_i \longmapsto 0$ under $K \otimes N \longrightarrow F \otimes N$, we can replace F by $R^{(J_0)}$ where

$$J_0 = \left\{ j \in J \;\middle|\; \underset{\text{for some } i}{j^{th} \text{ entry of } k_i \text{ is non-zero}} \right\} \text{ a finite set}$$

$\square$

(22.6) Lemma. Let $F_1, F_2, N \in R$-mod be such that $\forall\, K_\ell \hookrightarrow F_\ell$, the induced map $K_\ell \otimes N \longrightarrow F_\ell \otimes N$ is injective

then the same is true for $F_1 \oplus F_2$.

Pf.. Let $K \hookrightarrow F_1 \oplus F_2$. Set $K_1 = K \cap F_1$

$$K_2 = \text{Image of} \left\{ \begin{tikzcd} K \arrow[r, hook] & F_1 \oplus F_2 \arrow[d] \\ & F_2 \end{tikzcd} \right\}$$

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r] \arrow[d, hook] & K \arrow[r] \arrow[d, hook] & K_2 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & F_1 \oplus F_2 \arrow[r] & F_2 \arrow[r] & 0
\end{tikzcd}$$

Tensoring with N gives

$\otimes$ is right exact

$$\begin{tikzcd} & K_1 \otimes N \arrow[r] \arrow[d] & K \otimes N \arrow[r] \arrow[d] & K_2 \otimes N \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_1 \otimes N \arrow[r] & (F_1 \oplus F_2) \otimes N \arrow[r] & F_2 \otimes N \arrow[r] & 0 \end{tikzcd}$$

(injective by assumption)

(injective by assumption)

exact because split

$\Rightarrow$ (easy diagram chase; or snake lemma)

$$K \otimes N \longrightarrow F \otimes N = (F_1 \oplus F_2) \otimes N \text{ is injective}$$

□

(22.7) Cor.: $\mathbb{Q}$ is flat. More generally, for a P.I.D. R,
(of Thm 22.5)

flat <=> torsion free.