(23.0) Review of localization. Let R be a commutative ring $(1 \neq 0)$; and let $S \subset R \setminus \{0\}$, $1 \in S$, be a multiplicatively closed set.

$$S^{-1}R := S \times R \Big/ \begin{array}{l}(s_1, r_1) \sim (s_2, r_2) \\ \text{if } \exists\, t \in S \text{ s.t. } t(s_1 r_2 - s_2 r_1) = 0\end{array}$$

Notation: $\frac{r}{s}$ for $(s, r) \in S^{-1}R$

$S^{-1}R$ is a ring with
$$\begin{cases} \dfrac{r_1}{s_1} \cdot \dfrac{r_2}{s_2} = \dfrac{r_1 r_2}{s_1 s_2} \\[2ex] \dfrac{r_1}{s_1} + \dfrac{r_2}{s_2} = \dfrac{s_2 r_1 + s_1 \cdot r_2}{s_1 s_2} \end{cases}$$

$$j_S : \begin{array}{ccc} R & \longrightarrow & S^{-1}R \\ \Psi & & \Psi \\ r & \longmapsto & \frac{r}{1} \end{array}$$ is a ring homomorphism

We can view $S^{-1}(\cdot) : R\text{-mod} \longrightarrow S^{-1}R\text{-mod}$ as a functor which is same as $S^{-1}R \underset{R}{\otimes} (\cdot)$ (Homework 7 Problem R.)

Exercise: $S^{-1}(M \underset{R}{\otimes} N) \simeq S^{-1}M \underset{S^{-1}R}{\otimes} S^{-1}N$

$\left(\simeq (S^{-1}M) \underset{R}{\otimes} N\right)$ when we view an $S^{-1}R$-module as an R-module via $j_S : R \to S^{-1}R$.

(23.1) Recall that an ideal $\mathfrak{p} \subsetneq R$ is called <u>prime</u> if

$R/\mathfrak{p}$ is an integral domain (i.e. $xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p}$)

This condition is equivalent to $R \setminus \mathfrak{p}$ being a multiplicatively closed set.

$$R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1} R$$

<u>Proposition</u>. An R-module M is zero iff $M_{\mathfrak{p}} = 0$ $\forall$ prime $\mathfrak{p}$ iff $M_{\mathfrak{m}} = (0)$ $\forall$ maximal $\mathfrak{m}$.

<u>Pf</u>. The only non-trivial implication is $\underset{\forall \text{ max'l } \mathfrak{m}}{M_{\mathfrak{m}} = 0} \Rightarrow M = 0$.

So assume $M_{\mathfrak{m}} = (0)$ $\forall$ maximal ideal $\mathfrak{m}$. If $M \neq 0$, then we have some $x \in M$, $x \neq 0$. Thus $\mathfrak{a} = \mathrm{Ann}(x) = \{r \in R : rx = 0\}$ is a proper ideal, hence $\mathfrak{a} \subset \mathfrak{m}$ for some max'l ideal $\mathfrak{m}$.

Claim: $\frac{x}{1} \neq 0$ in $M_{\mathfrak{m}}$

Pf.: $\frac{x}{1} = 0 \iff \exists\, s \in R \setminus \mathfrak{m}$ s.t. $s.x = 0$

i.e. $\mathrm{Ann}(x) \not\subset \mathfrak{m}$; but that is not true. $\square$

$\square$

(23.2) As a consequence of Prop. (23.1) and Exercise (23.0) above, we get that

Proposition. For $N \in R$-mod, N is flat R-module iff $N_{\mathfrak{p}}$ is flat $R_{\mathfrak{p}}$-module $\forall$ prime ideal $\mathfrak{p} \subsetneq R$; iff $N_{\mathfrak{m}}$ is flat $R_{\mathfrak{m}}$-module $\forall$ max'l ideal $\mathfrak{m} \subsetneq R$.

(23.3). For local rings, we can prove the following necessary condition. Assume for now that R is a local ring (i.e. $\exists$ a unique maximal ideal), say $m \subsetneq R$ is its unique max'l ideal. Let $k = R/m$ be its residue field.

Proposition. – Let M be a flat R-module which admits a finite presentation. Then M is free.

Remarks. – (1) Finite presentation means that M is finitely generated, and can be generated by finitely many elements which have only finictely many relations. That is, there is an R-linear map between two free modules of finite rank $F_1 \xrightarrow{\pi} F_0$ s.t. $\mathrm{Coker}(\pi) = M$.

(2) We will only use $\mathrm{Tor}_1^R(k, M) = 0$, when Tor will be defined below.

Proof. One first proves that $\forall$ surjective R-linear maps $F \xrightarrow{\pi} M \to 0$, where F is free of finite rank, $K = \operatorname{Ker}(\pi)$ is finitely generated. To prove this, consider : (Remark 1 of page 3)

$$\begin{tikzcd} & F_2 \arrow[r] \arrow[d, "\phi"'] & F_1 \arrow[r] \arrow[d, "\psi"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r] & F \arrow[r, "\pi"'] & M \arrow[r] & 0 \end{tikzcd}$$

(F_1, F_2, F are free of finite rank)

- ψ exists because F_1 is projective.
- restricts to $F_2 \xrightarrow{\phi} K$, because $K = \operatorname{Ker}(\pi)$

Now we get that $K/\phi(F_2) \simeq F/\psi(F_1)$ (use Snake lemma, for example)

$\Rightarrow$ $K/\phi(F_2)$ is finitely generated, being quotient of a f.g. module F.

So $0 \to \phi(F_2) \to K \to K/\phi(F_2) \to 0$ exact, and $\phi(F_2), K/\phi(F_2)$ f.g.

$\Rightarrow$ K is f.g.

We will also need :

<u>Nakayama's Lemma.</u> $(R, \mathfrak{m})$: local ring. K : finitely gen. R-mod.

$$\mathfrak{m}K = K \quad \Rightarrow \quad K = \{0\}$$

<u>Pf.</u> Choose a set (finite) of generators of K, say $\{k_1, \dots, k_n\}$.

$$\mathfrak{m}K = K \Rightarrow \underset{(1 \le i,j \le n)}{\exists\, a_{ij} \in \mathfrak{m}} \quad \text{s.t.} \quad k_i = \sum_{j=1}^{n} a_{ij} k_j$$

i.e.
$$\left(Id_{n\times n} - (a_{ij})_{1 \le i,j \le n} \right) \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \det\left(Id_{n\times n} - (a_{ij}) \right) \in \operatorname{Ann}(K).$$

But $\det(\mathrm{Id} - (a_{ij})) \in 1 + m$ is hence is a unit. $\Rightarrow 1 \in \mathrm{Ann}(K) \Rightarrow K = \{0\}$. $\square$

Now we return to the proof of the proposition. Since M is finitely generated, let $\bar{x}_1, \ldots, \bar{x}_r \in M/_{mM}$ be a k-basis.

Let $x_1, \ldots, x_r \in M$ be representatives of $\{\bar{x}_j\}_{1 \le j \le r}$

Claim 1. $\{x_1, \ldots, x_r\}$ generates M.

Pf. of Claim 1: Let $M' \subset M$ be submodule gen. by $\{x_1, \ldots, x_r\}$. Let $K = M/_{M'}$. K is still finitely generated since M is. We show that $m(K) = K$, and hence by Nakayama's lemma; $K = 0$.

So, let $k \in K$. As $\{\bar{x}_1, \ldots, \bar{x}_r\}$ is k-basis of $M/_{mM}$, we get $\bar{k} = \sum_{j=1}^{r} \bar{\alpha}_j \bar{x}_j$ i.e. $k - \sum \alpha_j x_j \in mM$

(in $M/_{mM}$)

so, modulo M', $k \in mM$ proving the claim $\square$

Thus we have a surjective map of R-modules

$$F \to M \to 0 \qquad \text{s.t.} \qquad F/_{mF} \simeq M/_{mM}.$$

(F: free of finite rank)

$$\left(F \simeq R^{\oplus r} \longrightarrow M \quad \text{where } \{\bar{x}_i\} \text{ is a } k\text{-basis of } M/_{mM} \right)$$
$$\varepsilon_i \longmapsto x_i$$

Let K be the kernel of $F \longrightarrow M \to 0$, so that we have a short exact sequence $0 \to K \to F \to M \to 0$.

If M is flat, so will be K, since F is free, hence flat. (Homework 7, last problem).

K is finitely generated since, M is finitely presented and F is of finite rank.

Thus we get a snake diagram, using $m \otimes M \simeq mM$, $m \otimes K \simeq mK$,

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
mK \arrow[r] \arrow[d] & m \otimes F \arrow[r] \arrow[d] & mM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & K/_{mK} \arrow[r] & F/_{mF} \arrow[r, "\sim"] & M/_{mM} & \\
 & 0 \arrow[u, equal] & & &
\end{tikzcd}$$

($m \underset{R}{\otimes} -$ right exact)

assumption above

Thus we get $K/_{mK} = 0 \Rightarrow K = mK$, As it is also finitely generated, $K = 0$ by Nakayama's lemma and hence $M \simeq F$ is free $\square$

(23.4) Cor. Over a local Noetherian ring (R, m),

f.g. flat $\Leftrightarrow$ free.

Proof. We only need to recall that over a Noetherian ring R,

finitely generated $\Leftrightarrow$ Noetherian module.

defined via ascending chain condition which holds for submodules of a Noetherian module

[Review: For arbitrary comm ring R, M, an R-module is Noetherian if $\forall$ chain of submodules $M_1 \subset M_2 \subset \cdots$ $\exists$ $k > 0$ s.t. $M_k = M_{k+1} = \cdots$ } (ACC)

We have the following equivalent conditions: (a) M is a Noetherian R-module (b) Every non empty set of submodules of M has a max'l element (c) every submodule of M is finitely-generated.

Moreover if we have a short exact sequence of R-modules : $0 \to M' \to M \to M'' \to 0$, then M is Noetherian if, and only if M' and M'' are.

Now if R is a Noetherian ring, which is same as saying R is a Noetherian module over itself; every f.g. module is a quotient of a $R^{\oplus n}$ for some $n \in \mathbb{Z}_{\geq 1}$. $R^{\oplus n}$ is Noetherian module since R is, and hence so is M being its quotient. - end of Review].

(23.5) $(x, y) = M$ is not a flat $R = K[x, y]$-module (K : a field).

Let $\mathfrak{m} = (x, y) \subset R$ be a max'l ideal

If M were flat, so would be $M^{loc} = M_{\mathfrak{m}} \subset R_{\mathfrak{m}}$.

As it is finitely generated, we would have that M^{loc} is freely generated by 2 elements x & y.

But it is not true ($R_{\mathfrak{m}} \oplus R_{\mathfrak{m}} \xrightarrow{[x, y]} M^{loc}$ has non-trivial kernel).