Lecture 24

(24.0) Recall: R is a commutative ring with $1 \neq 0$. For $N \in R$-mod, we consider $- \otimes N : R\text{-mod} \longrightarrow R\text{-mod}$ as an additive, right-exact functor.

(24.1) Definition. Given $M, N \in R$-mod, let $P_\bullet$ be a projective resolution of M. Thus we have an exact sequence where each P_ℓ is projective R-module.

$$\cdots \to P_\ell \xrightarrow{d_\ell} P_{\ell-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \to 0$$

Define $\mathrm{Tor}_\ell^R(M,N) = \ell^{th}$ homology of the chain complex $P_\bullet \otimes N =$

$$\cdots \to P_1 \otimes N \xrightarrow{d_1 \otimes 1} P_0 \otimes N \to 0$$

$$= \frac{\mathrm{Ker}(d_\ell \otimes \mathrm{Id}_N)}{\mathrm{Im}(d_{\ell+1} \otimes \mathrm{Id}_N)}$$

(24.2) Properties: (1) $\mathrm{Tor}_0^R(M,N) = M \underset{R}{\otimes} N \; (= \mathrm{Tor}_0^R(N,M))$

(2) If N is projective $\mathrm{Tor}_i^R(M,N) = 0 \quad \forall\, M \in R$-mod

$[\, \mathrm{Tor}_i^R(M,N) = \mathrm{Tor}_i^R(N,M) \,]$

(3) $\mathrm{Tor}_\ell^R(-, N) : R\text{-mod} \longrightarrow R\text{-mod}$

$(\ell \geq 0)$ are additive functors with connecting morphisms: $\forall\ 0 \to M_1 \to M_2 \to M_3 \to 0$ exact sequence, we have

$$\begin{tikzcd}
 & \vdots & \\
\mathrm{Tor}_1(M_1,N) \arrow[r] & \mathrm{Tor}_1(M_2,N) \arrow[r] & \mathrm{Tor}_1(M_3,N) \arrow[dll, rounded corners] \\
M_1 \otimes N \arrow[r] & M_2 \otimes N \arrow[r] & M_3 \otimes N \arrow[r] & 0
\end{tikzcd}$$

long-exact sequence

→ Connecting morphisms are natural.

(24.3) Proposition. For an R-module N, the following are equivalent:

(1) ~~M~~ N is flat

(2) $\mathrm{Tor}_\ell^R(M, N) = 0 \quad \forall\, M \in R\text{-mod},\ \ell \geq 1$

(3) $\mathrm{Tor}_1^R(M, N) = 0 \quad \forall\, M \in R\text{-mod}$

(4) $\mathrm{Tor}_1^R(R/\mathfrak{a}, N) = 0 \quad \forall$ ideal $\mathfrak{a} \subset R$

Pf. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1). We have a short exact sequence

$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$. Tensoring with N yields

$$\cdots \quad \mathrm{Tor}_1^R(R;N) \to \mathrm{Tor}_1^R(R/\mathfrak{a};N) \to \mathfrak{a} \otimes N \to N$$
$$\to \frac{N}{\mathfrak{a}N} = (R/\mathfrak{a}) \otimes N \to 0$$

$$\mathrm{Tor}_1^R(R/\mathfrak{a};N) = 0 \implies \mathfrak{a} \otimes N \xrightarrow{\cong} \text{Kernel of } (N \to N/\mathfrak{a}N) = \mathfrak{a} \cdot N$$

Hence N is flat by Theorem (22.5) page 6.

(1) $\Rightarrow$ (2). Let F be a free R-module, together with a surjective map to N : $0 \to K \to F \to N \to 0$

Thus (assuming N is flat) K is flat. The long-exact sequence, and the fact that F is free, hence $\mathrm{Tor}_\ell^R(F,-) = 0$ $\forall \ell \geq 1$ ~~we get~~ yields:

$$\cdots \to \underbrace{\mathrm{Tor}_\ell(M,F)}_{0} \to \mathrm{Tor}_\ell(M,N) \xrightarrow{\cong} \mathrm{Tor}_{\ell-1}(M,K) \to \underbrace{\mathrm{Tor}_{\ell-1}(M,F)}_{0} \cdots$$

Thus we will be done by induction, except for the base case ($\ell = 1$).

Base case: $\cdots \; \underbrace{\mathrm{Tor}_1(M,F)}_{0 \text{ since } F \text{ is free}} \to \mathrm{Tor}_1(M,N) \to M\otimes K \to M\otimes F \to M\otimes N \to 0$

Now ~~$M\otimes$~~ $0 \to M\otimes K \to M\otimes F \to M\otimes N \to 0$ is exact because N is flat (Homework 7; Problem 6).

$\Rightarrow \mathrm{Tor}_1(M,N) = 0 \quad \forall\, M \in R\text{-mod}$

$\square$

(24.4) In the proof of Prop. 24.3 above, we used the following interpretation of $\mathrm{Tor}_1^R(R/\mathfrak{a}, N)$:

Lemma. -
$$\mathrm{Tor}_1^R(R/\mathfrak{a}, N) = \mathrm{Ker}\left(\mathfrak{a} \otimes_R N \longrightarrow \mathfrak{a}\cdot N\right)$$

Pf. - As $0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$ is exact

we get

$$\cdots \; \underbrace{\mathrm{Tor}_1(R,N)}_{0 \text{ as } R \text{ is free.}} \to \mathrm{Tor}_1(R/\mathfrak{a}, N) \to \mathfrak{a}\otimes N \to N \to N/\mathfrak{a}N \to 0$$

$$a\otimes n \mapsto a\cdot n$$

$\square$

(24.5) As remarked in §23.3, page 3, after the proposition, if $(R, \mathfrak{m})$ is a local ring and M is finitely presented, then M is flat $\iff \operatorname{Tor}_1^R(R/\mathfrak{m}, M) = 0$

(i.e. $\mathfrak{m} \otimes_R M \longrightarrow \mathfrak{m}M$ is an iso.)

Non-example. Take $R = K[x,y]_{(x,y)}$ (localized at max'l ideal (x,y)), where K is a field

i.e. an element of R is a fraction $\dfrac{p(x,y)}{q(x,y)}$, where $p(x,y), q(x,y) \in K[x,y]$ s.t. $q(0,0) \neq 0$.

$$M = K(x) = \left\{ \frac{a(x)}{b(x)} : a, b \in K[x] \text{ s.t. } b \neq 0 \right\}$$

R acts on M by setting $y = 0$, i.e.

$$\frac{p(x,y)}{q(x,y)} \cdot f(x) = \frac{p(x,0)}{q(x,0)} f(x) \in M$$

As M has torsion, it is not flat. But still

$\operatorname{Tor}_1^R(R/\mathfrak{m}, M) = 0$ - Exercise. - Hint - use Problem 3 of HW 7

(here $\mathfrak{m} \subset R$ is the unique max'l ideal $= (x,y) \subset K[x,y]_{(x,y)}$)

(24.6) Proposition (24.3) also implies that flat resolutions can be used instead of projective resolutions in order to compute $\mathrm{Tor}^R_\bullet(M,N)$.

Lemma. Let $0 \to K \to F \to M \to 0$ be a short exact seq. (of R-mod) when $F \in R\text{-mod}$ is a flat R-module. Then, $\forall N \in R\text{-mod}$

(1) $\mathrm{Tor}^R_1(M,N) = \text{Kernel of } (K\otimes N \to F\otimes N)$

(2) $\mathrm{Tor}^R_\ell(M,N) \simeq \mathrm{Tor}^R_{\ell-1}(K,N) \quad \forall\, \ell \geq 2$

Proof. – Obvious from the long exact sequence

$$\cdots \cancel{\mathrm{Tor}^R_\ell(F,N)}^{\,0} \to \mathrm{Tor}^R_\ell(M,N) \to \mathrm{Tor}^R_{\ell-1}(K,N) \to \cancel{\mathrm{Tor}^R_{\ell-1}(F,N)}^{\,0} \to \cdots$$

(by Prop 24.3 for F: flat)

$(\forall\, \ell \geq 2)$

and for $l = 1$:

$$\cancel{\mathrm{Tor}_1(F,N)}^{\,0} \to \mathrm{Tor}_1(M,N) \to K\otimes N \to F\otimes N \to M\otimes N \to 0$$

□

Cor. Let $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ be an exact sequence where each F_ℓ is flat $(\ell \geq 0)$. Then

$$\mathrm{Tor}^R_\ell(M,N) = \ell^{th} \text{ homology of the chain complex } \cdots \to F_1\otimes N \to F_0\otimes N \to 0$$

Proof. Hint: take $F = F_0$, $K = \mathrm{Ker}(F_0 \to M) = \mathrm{Im}(F_1 \to F_0)$

in the previous lemma, and note that

$$\cdots \to F_3 \to F_2 \to F_1 \to K \to 0 \quad \text{is again exact}$$

The lemma sets up an induction argument (in ℓ) to prove the assertion of the corollary.

$\square$

(24.7) Example. $R = \mathbb{Z}$, $M = \mathbb{Q}/\mathbb{Z}$, $N \in \mathcal{Ab}$ arbitrary

$$0 \to \mathbb{Z} \to \mathbb{Q} \to 0 \quad \text{is a flat resolution of } M$$
(not free)

$- \otimes N$: $\quad 0 \to N \longrightarrow N \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \to 0$

$$\Rightarrow \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, N) = \text{Kernel of } \begin{array}{ccc} N & \longrightarrow & N \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & x \otimes 1 \end{array}$$

Viewing $N \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ as $S^{-1}N$ when $S = \mathbb{Z} \setminus \{0\}$,

$$\frac{x}{1} = 0 \iff \exists\, s \in S \text{ s.t. } s.x = 0$$
$$\iff x \in N_{tor}.$$

$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, N) = N_{tor}.$$