Lecture 25 - Ext vs Tor

(25.0) Recall : in $\mathcal{A} = R$-mod (R is a commutative ring with $1 \neq 0$) we defined $\mathrm{Ext}_R^k(M,N)$, $\mathrm{Tor}_\ell^R(M,N)$ for $M, N \in R$-mod. ; $k, \ell \geq 0$. We will end this topic by saying a few words about functoriality of these constructions.

(25.1) $\mathrm{Ext}_R^k(A,B) \otimes \mathrm{Ext}_R^\ell(B,C) \to \mathrm{Ext}_R^{k+\ell}(A,C)$.

The composition map
$$\begin{aligned} \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) &\longrightarrow \mathrm{Hom}(A,C) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$
is by definition R-bilinear, and hence gives an R-linear map
$$\mathrm{Hom}_R(A,B) \otimes_R \mathrm{Hom}_R(B,C) \to \mathrm{Hom}_R(A,C)$$

This natural transformation (of appropriate functors - left as an exercise) extends to give R-linear maps

$$\mathrm{Ext}_R^k(A,B) \otimes_R \mathrm{Ext}_R^\ell(B,C) \to \mathrm{Ext}_R^{k+\ell}(A,C) \quad - (*)$$

Remarks. (1) One way to understand $(*)$ is to use the interpretation of elements of Ext as exact sequences.

For instance, (*) for $k=0$: $\mathrm{Hom}(A,B) \otimes \mathrm{Ext}^{l}(B,C) \to \mathrm{Ext}^{l}(A,C)$ comes from the fact that each $\mathrm{Ext}^{l}(-,C)$ is a contravariant functor.

(*) for $k=1$: $\xi \in \mathrm{Ext}^{1}(A,B) \leftrightarrow$ a short exact seq.

$$0 \to B \to X \to A \to 0$$

whose connecting hom gives a map $(\forall\, l \geq 0)$

$$\mathrm{Ext}^{l}(B,C) \longrightarrow \mathrm{Ext}^{l+1}(A,C)$$

– natural in short-exact sequences

(2) More conceptually, one needs to pick a projective resolution of A, $P_\bullet \to A \to 0$; an injective resolution of C, $0 \to C \to I^\bullet$, and consider $(\forall\, p,q \geq 0)$

$$\mathrm{Hom}(P_p, B) \times \mathrm{Hom}(B, I^q) \longrightarrow \mathrm{Hom}(P_p, I^q)$$

which will give the desired map. Here we will have to use that the total cohomology of the double complex $\{\mathrm{Hom}(P_p, I^q)\}$ computes $\mathrm{Ext}(A,C)$.

(25.2) For $A = B = C$, we get a (graded) product structure on $\mathrm{Ext}^\bullet(A,A) = \bigoplus_{k \geq 0} \mathrm{Ext}^k_R(A,A)$; analogous to the product structure on cohomology.

Tor has no such intrinsic structure (except for the obvious ones like $\mathrm{Tor}(-,N) = \mathrm{Tor}(N,-)$). It does admit an action of Ext :

(25.3) $$\mathrm{Ext}^k(A,B) \otimes \mathrm{Tor}_\ell(A,C) \to \mathrm{Tor}_{\ell-1}(B,C).$$

Again the natural map

$$\mathrm{Hom}(A,B) \times (A \otimes C) \longrightarrow B \otimes C$$
$$(\eta, a \otimes c) \longmapsto \eta(a) \otimes c$$

extends to R-linear maps

$$\mathrm{Ext}^k(A,B) \otimes \mathrm{Tor}_\ell(A,C) \to \mathrm{Tor}_{\ell-k}(B,C) \quad - (**)$$

For instance $k=0$: $\mathrm{Hom}(A,B) \times \mathrm{Tor}_\ell(A,C) \to \mathrm{Tor}_\ell(B,C)$ is saying that $\mathrm{Tor}_\ell(-,C)$ is a covariant functor (additive).

$k=1$. $\xi \in \mathrm{Ext}^1_R(A,B) \longleftrightarrow$ a short exact sequence

$$0 \to B \to X \to A \to 0$$

The associated long exact sequence of Tor functors includes as connecting morphisms

$$\mathrm{Tor}_\ell(A,C) \longrightarrow \mathrm{Tor}_{\ell-1}(B,C)$$

$=$ (∗∗) for $k=1$.

Thus $\mathrm{Tor}_\bullet(A,C) = \bigoplus \mathrm{Tor}_\ell(A,C)$ behaves as a module over $\mathrm{Ext}^\bullet(A,A)$.