

Lecture 40

①

(40.0) Let L/K be a Galois extension (i.e., normal & separable)

Let I be the index set for $\left\{ \begin{array}{l} \text{sub-}K\text{-extns.} \\ K \subset L_i \subset L \\ (i \in I) \end{array} \right. \text{ s.t. } L_i/K \text{ is finite \& Galois}$

Partial Order $i \leq j$ if $L_i \subset L_j$

Lemma $L = \bigcup_{i \in I} L_i$ ($= \varinjlim_{i \in I} L_i$)

Proof - As $\forall i \in I, L_i \hookrightarrow L$, we get, $\varinjlim L_i \xrightarrow{\varphi} L$

$\left[\begin{array}{c} \forall i \leq j, L_i \hookrightarrow L_j \\ \downarrow \quad \uparrow \\ L \end{array} \right. \text{ commutes}$

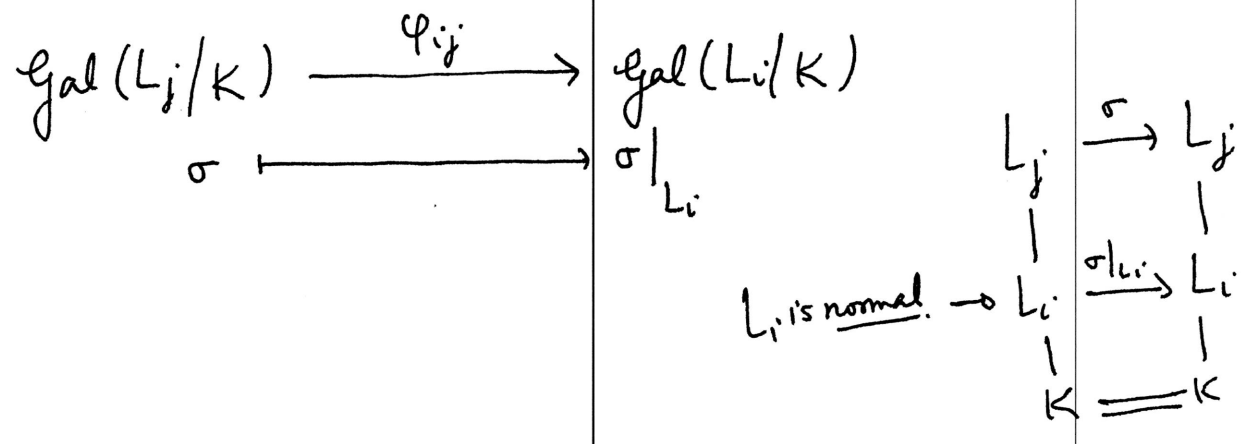
φ is injective since $\varphi|_{L_i} = L_i \hookrightarrow L$ is. Now $\forall \alpha \in L$,

let $L_j =$ splitting extn. of min. poly of $\alpha \in L$.

Then L_j/K is finite Galois and $\alpha \in L_j \Rightarrow \varphi$ is surjective. \square

(40.1) Now set $G_j = \text{gal}(L_j/K) \forall j \in I$. It is

finite and we use discrete topology on it. Moreover, $\forall i \leq j$ and



let $G = \text{Gal}(L/K)$. For each $i \in I$, $\sigma|_{L_i} \in \text{Gal}(L_i/K)$
 $\sigma \in G$

\Rightarrow We have a group hom. $G \longrightarrow \varprojlim_{i \in I} G_i$

It is surjective since L is Galois, and injective since if $\sigma \in G$,
 (prop. of normal extns.)
 $\sigma|_{L_i} = \text{id}_{L_i} \forall i \in I$, then, because $L = \varinjlim_{i \in I} L_i$, $\sigma = \text{id}$. \square

Thus $G \cong \varprojlim_{i \in I} G_i$ ($\text{Gal}(L/K) = \varprojlim_{i \in I} \text{Gal}(L_i/K)$)

The topology inherited on G has the following set of fund. nbd. of $e \in G$:

$$\mathcal{N}(e) = \left\{ N_i = \{ \sigma \in G \mid \sigma|_{L_i} = \text{id}_{L_i} \} \triangleleft G \right\}_{i \in I}$$

$$(\{1\} \longrightarrow N_i \longrightarrow G \longrightarrow G_i \longrightarrow \{1\})$$

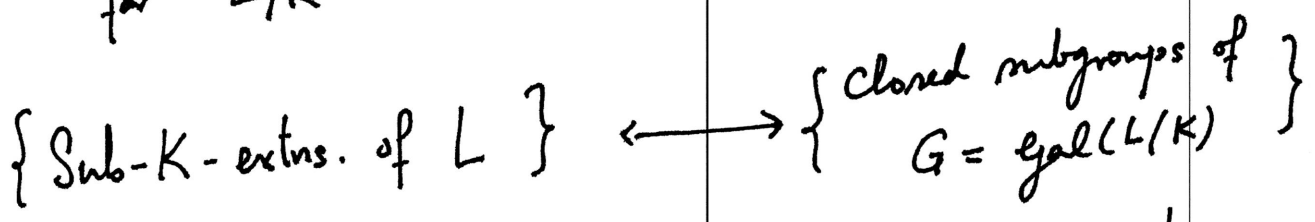
Note: The partially ordered set I is directed. That is, (I, \leq)

given $i, j \in I$ we can find $k \in I$ s.t. $i \leq k$ & $j \leq k$. Proof:

If L_i and L_j are corresponding extns., then L_i (resp. L_j) is the splitting extn. of some separable $p_i(x) \in K[x]$ (resp. $p_j(x)$).

As L itself is normal, $\exists L_k \subset L$ exists which is the splitting extn. of $\text{lcm}(p_i(x), p_j(x))$. So $L_i \subset L_k$ & $L_j \subset L_k$ as desired. \square

(40.2) Main Theorem We have the Galois correspondence for L/K - Galois extn. (ie $L^{\text{Gal}(L/K)} = K \equiv \text{sep. \& normal}$)



Notations. - Recall $\{L_i\}_{i \in I}$ = set of subextns. $\left. \begin{matrix} L \\ \cup \\ L_i \\ \cup \\ K \end{matrix} \right\}$ finite Galois

$$N(e) = \{ N_i = \text{Gal}(L/L_i) \triangleleft \text{Gal}(L/K) = G \}_{i \in I}$$

↑ fundamental system of rhd's of $e \in G$.

Proof. (\Rightarrow) Let $K \subset K' \subset L$ be a subextn.

We need to prove that $H = \text{Gal}(L/K') < G = \text{Gal}(L/K)$ is closed. We note that

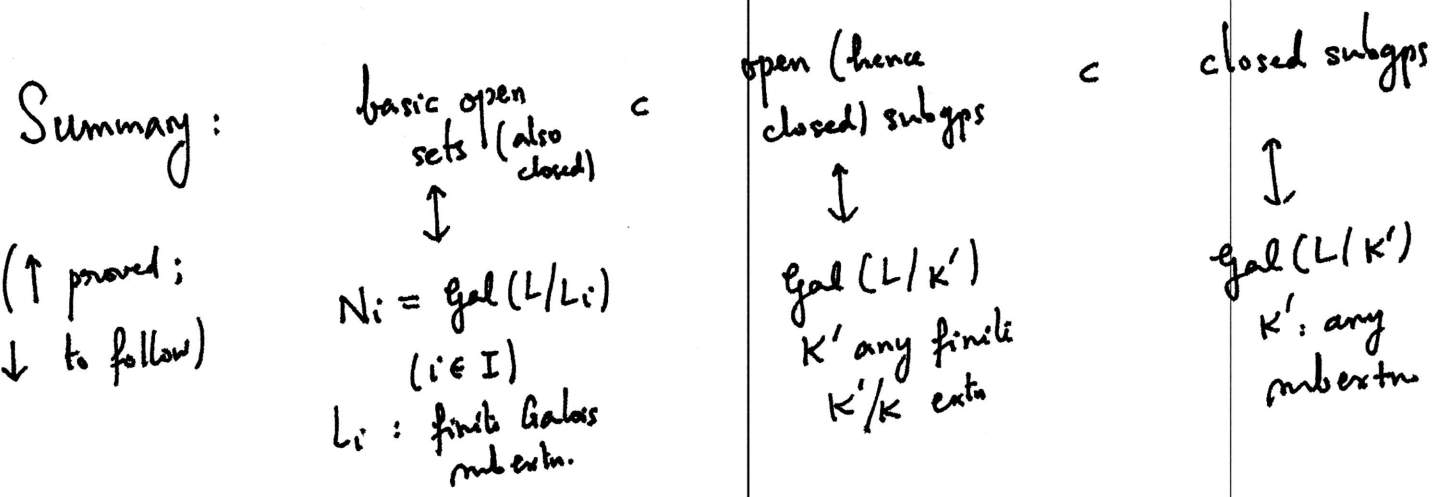
$$H = \text{Gal}(L/K') = \bigcap_{\alpha \in K'} \text{Gal}(L/K(\alpha))$$

Thus it is enough to prove that $\text{Gal}(L/K(\alpha))$ is closed. We will prove that $H_\alpha = \text{Gal}(L/K(\alpha))$ is open, hence closed as it is a subgroup.

Now take $L_i =$ splitting extn. of min'l poly of $\alpha \subset L$

$$H_\alpha = \text{Gal}(L/K_\alpha) \supset \text{Gal}(L/L_i) = N_i \in \mathcal{N}(e).$$

so $H_\alpha = \bigcup_{\bar{h} \in H_\alpha/N_i} h N_i =$ union of open sets, hence open

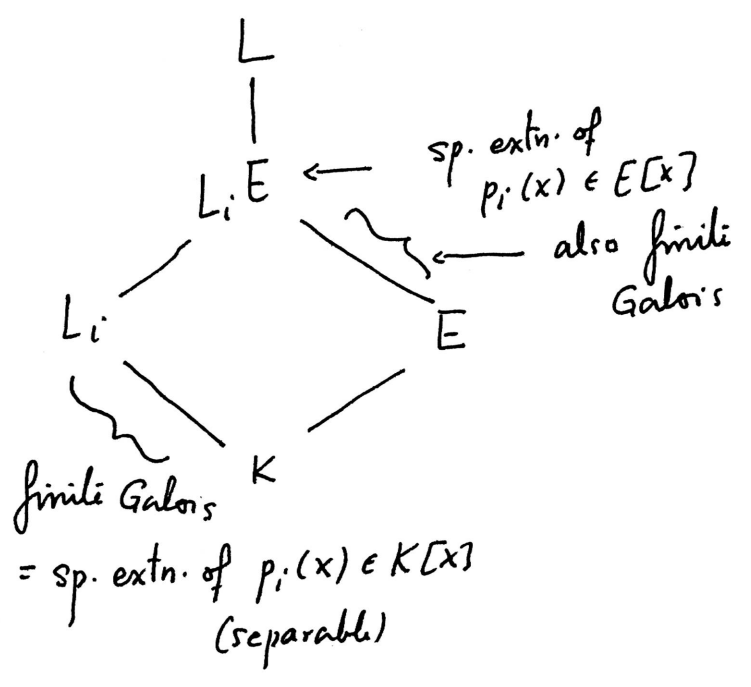


Conversely. Let $H < G = \text{Gal}(L/K)$ be a closed subgroup. Take $E = L^H$. We are to show: $\text{Gal}(L/E) = H$ and the other way.

(40.3) Krull's Theorem. - Let $H < G$ be a subgroup. Then H is dense in $\text{Gal}(L/L^H)$ (we already proved, it is closed).

Proof. Set $E = L^H$. We need to prove that

$$\forall \sigma \in \text{Gal}(L/E), \quad \forall i \in I, \quad \sigma N_i \cap H \neq \emptyset.$$



$$\begin{array}{ccc} H & & \\ \wedge & & \\ \text{Gal}(L/E) & \xrightarrow[\text{restriction}]{\rho} & \text{Gal}(L_i E/E) \\ \psi & & \\ \sigma & & \end{array}$$

$$\begin{aligned} E &= L^H \\ \Rightarrow (L_i E)^{P(H)} &= E \\ \Rightarrow \rho(H) &= \text{Gal}(L_i E/E) \\ \Rightarrow \rho(\sigma) &\in \rho(H) \end{aligned}$$

so $\rho(\sigma) = \rho(h)$ for some $h \in H$

$$\Rightarrow \sigma^{-1} h \in \text{Ker}(\rho) = \text{Gal}(L/L_i E) \subset \text{Gal}(L/L_i) = N_i$$

$$\Rightarrow \sigma N_i \cap H \neq \emptyset$$

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