Lecture 40

(40.0) Let L/K be a Galois extension (i.e, normal & separable)

Let I be the index set for $\left\{ \begin{array}{l} \text{sub-}K\text{-extns.} \\ K \subset L_i \subset L \\ (i \in I) \end{array} \text{ s.t } \begin{array}{l} L_i/K \text{ is} \\ \text{finite \& Galois} \end{array} \right\}$

Partial Order $i \leq j$ if $L_i \subset L_j$

Lemma $L = \bigcup_{i \in I} L_i \quad \left(= \varinjlim_{i \in I} L_i \right)$

Proof – As $\forall i \in I$, $L_i \hookrightarrow L$, we get, $\varinjlim L_i \xrightarrow{\varphi} L$

$$\left[\& \ \forall i \leq j, \quad \begin{tikzcd} L_i \arrow[r, hook] \arrow[dr, hook] & L_j \arrow[d, hook] \\ & L \end{tikzcd} \text{ commutes} \right]$$

φ is injective since $\varphi\big|_{L_i} = L_i \hookrightarrow L$ is. Now $\forall \alpha \in L$,

let L_j = splitting extn. of min'l poly of α $\subset L$.

Then L_j/K is finite Galois and $\alpha \in L_j \Rightarrow \varphi$ is surjective. □

(40.1) Now set $G_j = \mathcal{G}al(L_j/K)$ $\forall j \in I$. It is

finite and we use discrete topology on it. Moreover, $\forall i \leq j$ and

$$\begin{array}{ccc} Gal(L_j/K) & \xrightarrow{\varphi_{ij}} & Gal(L_i/K) \\ \sigma & \longmapsto & \sigma\big|_{L_i} \end{array}$$

$$\begin{tikzcd} & L_j \arrow[r, "\sigma"] \arrow[d, dash] & L_j \arrow[d, dash] \\ L_i \text{ is } \underline{\text{normal}}. \arrow[r] & L_i \arrow[r, "\sigma|_{L_i}"] \arrow[d, dash] & L_i \arrow[d, dash] \\ & K \arrow[r, equal] & K \end{tikzcd}$$

Let $\underset{\substack{\cup\!\!\!| \\ \sigma}}{G} = Gal(L/K)$. For each $i \in I$, $\sigma\big|_{L_i} \in Gal(L_i/K)$

$\Rightarrow$ We have a group hom. $G \longrightarrow \varprojlim_{i \in I} G_i$

It is surjective since L is Galois (prop. of normal extns.), and injective since if $\sigma \in G$, $\sigma\big|_{L_i} = Id_{L_i}$ $\forall i \in I$, then, because $L = \varinjlim_{i \in I} L_i$, $\sigma = Id$. $\square$

Thus $G \simeq \varprojlim_{i \in I} G_i$ $\left(Gal(L/K) = \varprojlim_{i \in I} Gal(L_i/K) \right)$

The topology inherited on G has the following set of fund. nhds. of $e \in G$:

$$\mathcal{N}(e) = \left\{ N_i = \{\sigma \in G \mid \sigma|_{L_i} = Id_{L_i}\} \lhd G \right\}_{i \in I}$$

$$\left(\{1\} \longrightarrow N_i \longrightarrow G \longrightarrow G_i \longrightarrow \{1\} \right)$$

Note: The partially ordered set $(I, \leq)$ is directed. That is, given $i, j \in I$ we can find $k \in I$ s.t. $i \leq k$ & $j \leq k$. Proof: If L_i and L_j are corresponding extns., then L_i (resp. L_j) is the splitting extn. of some separable $p_i(x) \in K[x]$ (resp. $p_j(x)$). As L itself is normal, $\exists$ $L_k \subset L$ exists which is the splitting extn. of $\mathrm{lcm}(p_i(x), p_j(x))$. So $L_i \subset L_k$, $L_j \subset L_k$ as desired. □

(40.2) Main Theorem. We have the Galois correspondence for L/K - Galois extn. (i.e. $L^{gal(L/K)} = K \equiv$ sep. & normal)

$$\{\text{Sub-}K\text{-extns. of } L\} \longleftrightarrow \left\{\begin{array}{c}\text{Closed subgroups of}\\ G = gal(L/K)\end{array}\right\}$$

Notations. – Recall $\{L_i\}_{i \in I}$ = set of subextns. $L \supset L_i \supset K$ } finite Galois

$$\mathcal{N}(e) = \{N_i = gal(L/L_i) \triangleleft gal(L/K) = G\}_{i \in I}$$

fundamental system of nbds of $e \in G$.

Proof. ($\Rightarrow$) Let $K \subset K' \subset L$ be a subextn.

We need to prove that $H = Gal(L/K') < G = Gal(L/K)$ is closed. We note that

$$H = Gal(L/K') = \bigcap_{\alpha \in K'} Gal(L/K(\alpha))$$

Thus it is enough to prove that $Gal(L/K(\alpha))$ is closed.

We will prove that $H_\alpha = Gal(L/K(\alpha))$ is open, hence closed as it is a subgp.

Now take L_i = splitting extn. of min'l poly of α $\subset L$

$H_\alpha = Gal(L/K_\alpha) \supset Gal(L/L_i) = N_i \in \mathcal{N}(e)$.

so $H_\alpha = \bigcup_{\bar{h} \in H_\alpha/N_i} hN_i$ = union of open sets, hence open

Summary:

	basic open sets (also closed)	$\subset$	open (hence closed) subgps	$\subset$	closed subgps
($\uparrow$ proved; $\downarrow$ to follow)	$\updownarrow$		$\updownarrow$		$\updownarrow$
	$N_i = Gal(L/L_i)$ $(i \in I)$ L_i : finite Galois subextn.		$Gal(L/K')$ K' any finite K'/K extn		$Gal(L/K')$ K': any subextn

Conversely. Let $H < G = Gal(L/K)$ be a closed subgroup.

Take $E = L^H$. We are to show: $Gal(L/E) = H$. and the other way.

(40.3) Krull's Theorem.— Let $H < G$ be a subgroup. Then H is dense in $\mathrm{Gal}(L/L^H)$ (we already proved, it is closed).

Proof. Set $E = L^H$. We need to prove that

$\forall \sigma \in \mathrm{Gal}(L/E)$, $\forall i \in I$, $\quad \sigma N_i \cap H \neq \phi$.

$$\begin{tikzcd}
 & L \arrow[d, dash] & \\
 & L_iE \arrow[dl, dash] \arrow[dr, dash] & \\
L_i \arrow[dr, dash] & & E \arrow[dl, dash] \\
 & K &
\end{tikzcd}$$

L_iE ← sp. extn. of $p_i(x) \in E[x]$

L_iE / E: also finite Galois

L_i / K: finite Galois = sp. extn. of $p_i(x) \in K[x]$ (separable)

$$\begin{array}{ccc}
H & & \\
\wedge & & \\
\mathrm{Gal}(L/E) & \xrightarrow[\text{restriction}]{\rho} & \mathrm{Gal}(L_iE/E) \\
\Psi & & \\
\sigma & &
\end{array}$$

$$E = L^H$$

$$\Rightarrow (L_iE)^{\rho(H)} = E$$

$$\Rightarrow \rho(H) = \mathrm{Gal}(L_iE/E)$$

$$\Rightarrow \rho(\sigma) \in \rho(H)$$

So $\rho(\sigma) = \rho(h)$ for some $h \in H$

$\Rightarrow \sigma^{-1}h \in \mathrm{Ker}(\rho) = \mathrm{Gal}(L/L_iE) \subset \mathrm{Gal}(L/L_i) = N_i$

$\Rightarrow \sigma N_i \cap H \neq \phi$ $\square$