

Optional Reading A

Axioms in abelian categories après Grothendieck.

(A.0) Let \mathcal{A} be an abelian category. Recall that it means

\mathcal{A} is an additive category $\left\{ \begin{array}{l} \text{Hom}_{\mathcal{A}}(X, Y) \text{ is an abelian group } \forall X, Y \in \mathcal{A} \\ \text{Compositions are } \mathbb{Z}\text{-bilinear} \\ \text{Finite direct sums (= products) \& } \mathcal{O}_{\mathcal{A}} \text{ (zero object) exist.} \end{array} \right.$

(AB1) $\text{Ker}(f)$ and $\text{Coker}(f)$ exist \forall morphism $f \in \text{Hom}_{\mathcal{A}}(X, Y)$

(AB2) $\forall f: X \rightarrow Y$ in \mathcal{A} , the induced morphism

$$\bar{f}: \text{Coim}(f) \longrightarrow \text{Im}(f) \text{ is an isomorphism.}$$

The following axioms on an abelian category are often imposed, and explicitly specified for results presented in this appendix.

(AB3) Arbitrary direct sums exist in \mathcal{A} .

(AB4) (AB3) & $\bigoplus_{i \in I} : \mathcal{A}^I \longrightarrow \mathcal{A}$ is exact

As we will see later, (AB3) $\Rightarrow \lim_{(I, \leq)}$ exist in \mathcal{A} and

is a right exact functor.

(AB5) (AB3) and $\lim_{(I, \leq)} : \mathcal{A}^{(I, \leq)} \longrightarrow \mathcal{A}$ is an exact functor

More explicitly, \lim_{\rightarrow} (injective morphisms) is injective.

The dual axioms are often denoted by *

(AB3*) Arbitrary direct products exist.

(AB4*) $\prod_{i \in I} A^I \rightarrow A$ is exact.

(AB5*) \varprojlim is an exact functor (i.e. $\varprojlim \text{inj}$ is inj.).

(A.1) Lemma. - Let \mathcal{A} be an abelian category satisfying (AB3).

Then $\varinjlim_{(I, \leq)}$ exist in \mathcal{A} , and is a right exact functor.

Proof. Let (I, \leq) be a preordered set and let

$\mathcal{X} = (\{X_i\}_{i \in I}, \{\psi_{ji}: X_i \rightarrow X_j\}_{i \leq j})$ be a direct system.

Let $S = \bigoplus_{i \in I} X_i$ (exists by (AB3)). For every $i \leq j$,

consider morphisms

$X_i \rightarrow X_i$	$: \text{id}_{X_i}$
$X_i \rightarrow X_j$	$: -\psi_{ji}$
$X_i \rightarrow X_k$	$: 0 \text{ if } k \neq i \text{ or } j$

(fixed $i \in I$)

Composing with natural morphisms $i_k: X_k \rightarrow S$, we get

$$X_i \xrightarrow{u_{ji}} S \quad \forall i \leq j$$

By defn. of the direct sum, we get a morphism

$$\bigoplus_{i \in I} \left(\bigoplus_{j \geq i} X_i \right) \xrightarrow{v} S \quad \text{Let } X = \text{Coker}(v)$$

Claim. - X , together with $X_i \xrightarrow{i_i} S \twoheadrightarrow X$
is the direct limit of \mathcal{X} .

Proof. - We need to check that $\forall Z \in \mathcal{A}$ and $X_i \xrightarrow{f_i} Z (i \in I)$
s.t. $X_i \xrightarrow{\psi_{ji}} X_j$ commutes, $\exists!$ morphism $X \xrightarrow{g} Z$ s.t.
 $f_i \downarrow \swarrow \nwarrow \uparrow$
 Z f_j $g \circ i_i = f_i \forall i \in I$.

We get a unique morphism $S \xrightarrow{\tilde{g}} Z$ s.t. $\tilde{g} \circ i_i = f_i \forall i \in I$
from the defn. of the direct sum. Moreover
 $\tilde{g} \circ u_{ji} = 0 \forall i \leq j \implies \tilde{g} \circ v = 0 \implies \tilde{g}$
factors through $\text{Coker}(v) = X$. □

Right exactness. Let $0 \rightarrow \mathcal{X}' \xrightarrow{\{u_i\}} \mathcal{X} \xrightarrow{\{v_i\}} \mathcal{X}'' \rightarrow 0$ be a short exact
sequence of direct systems. We want to show that the induced
sequence of direct limits $X' \xrightarrow{u} X \xrightarrow{v} X'' \rightarrow 0$ is exact.

That is, $\forall Z \in \mathcal{A}$ $\rightarrow 0 \rightarrow \text{Hom}_A(X'', Z) \xrightarrow{-\circ v} \text{Hom}_A(X, Z) \xrightarrow{-\circ u} \text{Hom}_A(X', Z)$
is exact [Mid Term 1].

$$0 \rightarrow \prod_{i \in I} \text{Hom}(X''_i, Z) \xrightarrow{\{-\circ v_i\}} \prod_{i \in I} \text{Hom}(X_i, Z) \xrightarrow{\{-\circ u_i\}} \prod_{i \in I} \text{Hom}(X'_i, Z)$$

But the lower sequence is exact (Homework 4). let us use
it to show that $\text{Ker}(-\circ u) \subset \text{Im}(-\circ v)$, for instance.

If $f \in \text{Hom}_A(X, Z)$ is s.t. $f \circ u = 0$, then, f comes from

$$f_i : X_i \rightarrow Z \quad \text{s.t.} \quad f_j \psi_{ji} = f_i \quad \forall i \leq j.$$

and $f_i \circ u_i = 0 \quad \forall i \in I$. The exactness of the bottom seq. implies $f_i = f_i'' \circ v_i$ for (unique) $f_i'' \in \text{Hom}(X_i'', Z)$. It ($\forall i \in I$).

remains to show that $\{f_i''\}_{i \in I}$ comes from $\text{Hom}(X'', Z)$, i.e.,

$$f_j'' \psi_{ji}'' = f_i'' \quad \forall i \leq j. \quad \text{But} \quad f_j'' \psi_{ji}'' \circ v_i = f_j'' \circ v_j \psi_{ji} \quad (\{v_i\} \text{ is a morphism of direct sys.})$$
$$= f_j \psi_{ji} = f_i = f_i'' \circ v_i$$

As v_i is surjective, we get $f_j'' \psi_{ji}'' = f_i''$ as required. \square

(A.2) Generator. An object $U \in A$ is said to be a generator if \forall injective morphism, which is not an isomorphism $A \xrightarrow{i} B$, $\exists f : U \rightarrow B$ which does not factor through A (i.e. there is no $\bar{f} : U \rightarrow A$ s.t. $f = i \circ \bar{f}$).

Equivalently, $\text{Hom}(U, A) \xrightarrow{i \circ -} \text{Hom}(U, B)$ is not a bijection.

Lemma. - If a generator exists in A , then for every $B \in A$

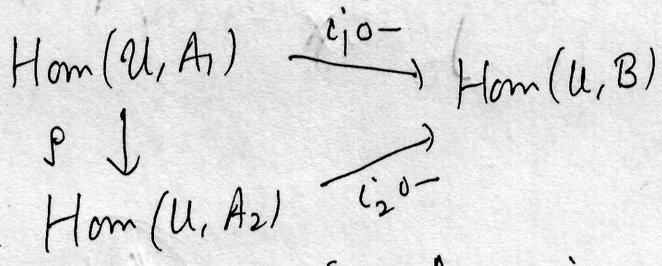
Subobjects of B is a set.
injective morphisms into B , up to isomorphism

Proof. We claim that $A \xrightarrow{i} B \mapsto (i_0) (\text{Hom}(U, A)) \subset \text{Hom}(U, B)$

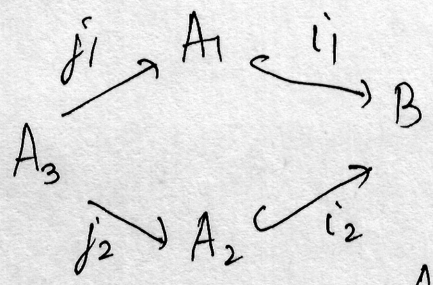
is injective (thus the class of subobjects of B is a subset of the power set of $\text{Hom}(U, B)$; hence a set).

Thus, assume $A_1 \xrightarrow{i_1} B$ and $A_2 \xrightarrow{i_2} B$ are such that there is a bijection

(T.S.) $A_1 \cong A_2$.



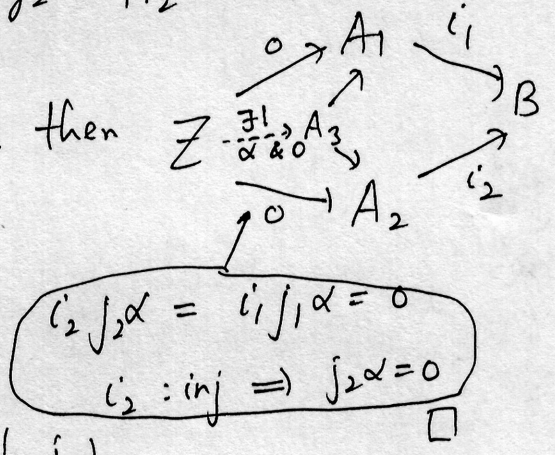
Consider the pull-back diagram



Claim 1. j_1 and j_2 are injective.

Pf. (for j_1 ; j_2 is similar)

if $Z \xrightarrow{\alpha} A_3$ is st. $j_1 \alpha = 0$, then $Z \xrightarrow{\frac{j_1 \alpha}{\alpha} \rightarrow 0} A_3$
 by uniqueness of $Z \rightarrow A_3$, $\alpha = 0$



Claim 2. j_1 is an isomorphism. (similarly j_2).

Pf. If not, we can find $f: U \rightarrow A_1$ which is not of the form $j_1 \bar{f}$ for any $\bar{f}: U \rightarrow A_2$. But there is $p(f): U \rightarrow A_2$ st. $i_2 p(f) = i_1 f \Rightarrow$ by defn. of pull-back that $f = j_1 \bar{f} = j_2 \bar{f}$ for a unique \bar{f} . Contradiction!

(A.3) Baer's Criterion for injectivity. -

(6)

Prop. Again A is abelian, (AB5) and has generator U .

$$M \in \mathcal{A} \text{ is injective} \Leftrightarrow \forall V \xrightarrow{i} U, \text{ and } \begin{array}{c} V \\ \downarrow f \\ M \end{array}, \exists$$

a lift $\tilde{f}: U \rightarrow M$ st. $\tilde{f} \circ i = f$.

Pf \Rightarrow obvious. Conversely, let there be given an injective

morphism $A \xrightarrow{i} B$ and an arbitrary $g: A \rightarrow M$.

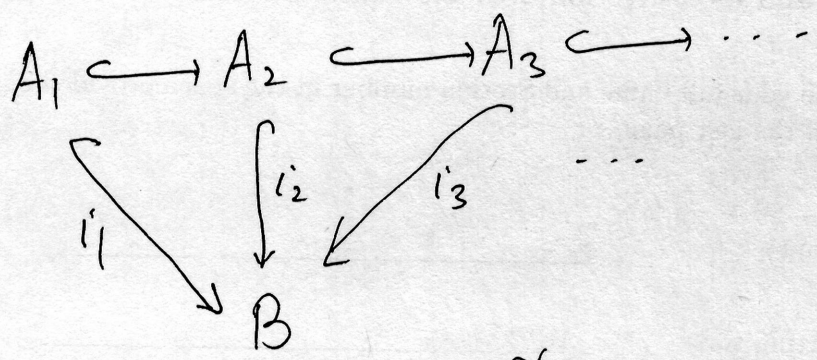
$\mathcal{P} =$ set of all $\begin{array}{c} A_1 \xrightarrow{i_1} B \\ \downarrow g_1 \\ M \end{array}$ st. $\begin{array}{c} A \xrightarrow{i} A_1 \\ g \downarrow \swarrow g_1 \\ M \end{array}$ commutes

by lemma A-2.

(up to iso. of course)

and $\begin{array}{c} A \xrightarrow{i} B \\ \downarrow \\ A_1 \end{array}$ commutes

Given a chain in \mathcal{P} (which is partially ordered by extn.)



we get

$$\tilde{A} = \varinjlim_{k \geq 1} A_k \xrightarrow{\tilde{i}} B$$

still injective by (AB5)

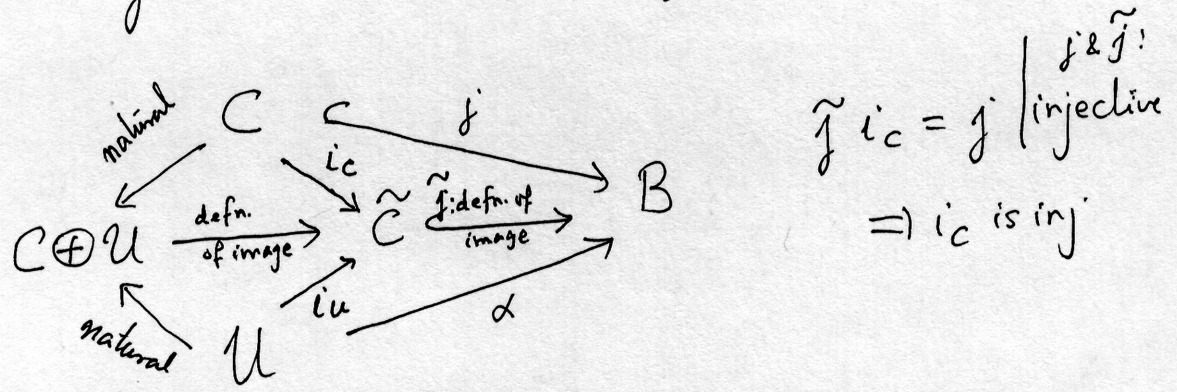
$$\tilde{g} = \varinjlim g_i \downarrow \\ M$$

Now let $C \xrightarrow{j} B$ be a max'l elt from \mathcal{P} .
 $\tilde{j} \downarrow$
 M

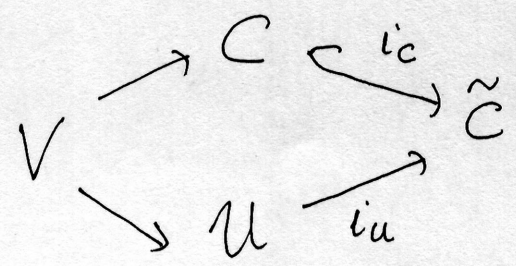
Claim: $C \xrightarrow{j} B$ is an iso.

Pf. If not, we can find $\alpha: U \rightarrow B$ which cannot be written as $j \circ \bar{\alpha}$. Let $\tilde{C} = \text{Image}(C \oplus U \rightarrow B)$

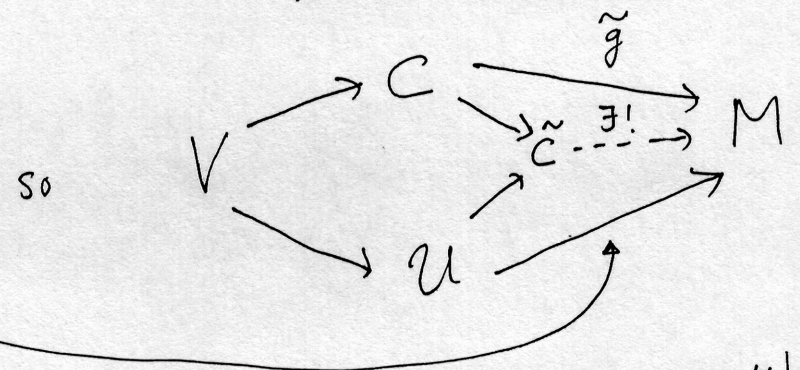
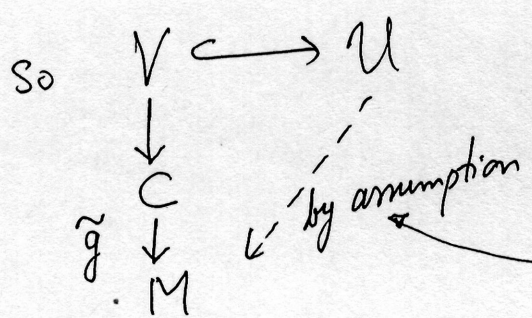
Pictorially



Again pull-back



Exercise: $V \rightarrow U$ is injective (because i_c is)
 (Hint: see page 5)



(see Lemma 12.4 page 4)

Thus by maximality of C , i_c is an iso.

$\Rightarrow \alpha = j \circ i_c^{-1} \circ i_u$ contradiction! \square