Optional Reading A

Axioms in abelian categories après Grothendieck.

(A.0) Let $\mathcal{A}$ be an abelian category. Recall that it means

$\mathcal{A}$ is an additive category $\begin{cases} \mathrm{Hom}_{\mathcal{A}}(X,Y) \text{ is an abelian group } \forall X,Y \in \mathcal{A} \\ \text{Compositions are } \mathbb{Z}\text{-bilinear} \\ \text{Finite direct sums (= products) \& } 0_{\mathcal{A}} \text{ (zero object) exist.} \end{cases}$

(AB1) $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ exist $\forall$ morphism $f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$

(AB2) $\forall\, f: X \to Y$ in $\mathcal{A}$, the induced morphism

$$\bar{f}: \mathrm{Coim}(f) \longrightarrow \mathrm{Im}(f) \text{ is an isomorphism.}$$

The following axioms on an abelian category are often imposed, and explicitly specified for results presented in this appendix.

(AB3) Arbitrary direct sums exist in $\mathcal{A}$.

(AB4) (AB3) & $\bigoplus_{i \in I} : \mathcal{A}^I \longrightarrow \mathcal{A}$ is exact

As we will see later, (AB3) $\Rightarrow$ $\varinjlim_{(I,\leq)}$ exist in $\mathcal{A}$ and is a right exact functor.

(AB5) (AB3) and $\varinjlim_{(I,\leq)} : \mathcal{A}^{(I,\leq)} \longrightarrow \mathcal{A}$ is an exact functor

More explicitly, $\varinjlim$ (injective morphisms) is injective.

The dual axioms are often denoted by *

(AB3*) Arbitrary direct products exist.

(AB4*) $\prod_{i\in I} : \mathcal{A}^I \longrightarrow \mathcal{A}$ is exact.

(AB5*) $\varprojlim$ is an exact functor (i.e. $\varprojlim$ surj is surj).

(A.1) Lemma.– Let $\mathcal{A}$ be an abelian category satisfying (AB3). Then $\varinjlim_{(I,\leq)}$ exist in $\mathcal{A}$, and is a right exact functor.

Proof. Let $(I, \leq)$ be a preordered set and let

$$\mathcal{X} = \left(\{X_i\}_{i\in I}, \{\psi_{ji} : X_i \to X_j\}_{i\leq j} \right)$$ be a direct system.

Let $S = \bigoplus_{i\in I} X_i$ (exists by (AB3)). For every $i \leq j$,

consider morphisms (fixed $i \in I$)

$$X_i \longrightarrow X_i : Id_{X_i}$$
$$X_i \longrightarrow X_j : -\psi_{ji}$$
$$X_i \longrightarrow X_k : 0 \text{ if } k \neq i \text{ or } j$$

Composing with natural morphisms $i_\ell : X_\ell \longrightarrow S$, we get

$$X_i \xrightarrow{u_{ji}} S \qquad \forall\, i \leq j$$

By defn. of the direct sum, we get a morphism

$$\bigoplus_{i\in I} \left(\bigoplus_{j\geq i} X_i \right) \xrightarrow{v} S .$$ Let $X = \mathrm{Coker}(v)$

Claim.— X, together with
$$\begin{tikzcd} X_\ell \arrow[r, "i_\ell"] \arrow[rr, bend right, "d_\ell"'] & S \arrow[r, two heads] & X \end{tikzcd}$$
is the direct limit of $\mathcal{X}$.

Proof.— We need to check that $\forall\, Z \in \mathcal{A}$ and $X_\ell \xrightarrow{f_\ell} Z$ $(\ell \in I)$ s.t.
$$\begin{tikzcd} X_i \arrow[rr, "\varphi_{ji}"] \arrow[dr, "f_i"'] & & X_j \arrow[dl, "f_j"] \\ & Z & \end{tikzcd}$$
commutes, $\exists!$ morphism $X \xrightarrow{g} Z$ s.t. $g \circ d_i = f_i \ \forall i \in I$.

We get a unique morphism $S \xrightarrow{\tilde{g}} Z$ s.t. $\tilde{g} \circ i_\ell = f_\ell \ \forall \ell \in I$ from the defn. of the direct sum. Moreover
$$\tilde{g} \circ u_{ji} = 0 \quad \forall\, i \leq j. \Rightarrow \tilde{g}\, v = 0 \Rightarrow \tilde{g}$$
factors through $\operatorname{Coker}(v) = X$. □

Right exactness. Let $0 \to \mathcal{X}' \xrightarrow{\{u_i\}} \mathcal{X} \xrightarrow{\{v_i\}} \mathcal{X}'' \to 0$ be a short exact sequence of direct systems., We want to show that the induced sequence of direct limits $X' \xrightarrow{u} X \xrightarrow{v} X'' \to 0$ is exact.

That is, $\forall\, Z \in \mathcal{A}$,
$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\mathcal{A}}(X'', Z) \arrow[r, "-\circ v"] \arrow[d, "\wr"] & \operatorname{Hom}_{\mathcal{A}}(X, Z) \arrow[r, "-\circ u"] \arrow[d, "\wr"] & \operatorname{Hom}_{\mathcal{A}}(X', Z) \arrow[d, "\wr"] \\ 0 \arrow[r] & \prod_{i \in I} \operatorname{Hom}(X_i'', Z) \arrow[r, "\{-\circ v_i\}"'] & \prod_{i \in I} \operatorname{Hom}(X_i, Z) \arrow[r, "\{-\circ u_i\}"'] & \prod_{i \in I} \operatorname{Hom}(X_i', Z) \end{tikzcd}$$
is exact [Mid Term 1].

But the lower sequence is exact (Homework 4). Let us use it to show that $\operatorname{Ker}(-\circ u) \subset \operatorname{Im}(-\circ v)$, for instance.

If $f \in \operatorname{Hom}_{\mathcal{A}}(X, Z)$ is s.t. $f \circ u = 0$, then, f comes from

$f_i : X_i \to Z$ s.t. $f_j \psi_{ji} = f_i \ \forall i \le j$.

and $f_i \circ u_i = 0 \ \forall i \in I$. The exactness of the bottom seq. implies $f_i = f_i'' v_i$ for (unique) $f_i'' \in Hom(X_i'', Z)$. It ($\forall i \in I$).

remains to show that $\{f_i''\}_{i \in I}$ comes from $Hom(X'', Z)$, i.e.,

$f_j'' \psi_{ji}'' = f_i'' \ \forall i \le j$. But $f_j'' \psi_{ji}'' v_i = f_j'' v_j \psi_{ji}$ ($\{v_i\}$ is a morphism of direct sys.)

$$= f_j \psi_{ji} = f_i = f_i'' v_i$$

As v_i is surjective, we get $f_j'' \psi_{ji}'' = f_i''$ as required. □

(A.2) Generator. An object $\mathcal{U} \in \mathcal{A}$ is said to be a generator if $\forall$ injective morphism, which is not an isomorphism $A \xrightarrow{i} B$, $\exists f : \mathcal{U} \to B$ which does not factor through A (i.e. there is no $\bar{f} : \mathcal{U} \to A$ s.t. $f = i \circ \bar{f}$).

Equivalently, $Hom(\mathcal{U}, A) \xrightarrow{i \circ -} Hom(\mathcal{U}, B)$ is not a bijection.

Lemma. – If a generator exists in $\mathcal{A}$, then for every $B \in \mathcal{A}$ $\boxed{\text{Subobjects of } B}$ is a set.

injective morphisms into B, up to isomorphism

Proof. We claim that $A \overset{i}{\hookrightarrow} B \longmapsto (i\circ -)(\mathrm{Hom}(U,A)) \subset \mathrm{Hom}(U,B)$

is injective (thus the class of subobjects of B is a subclass of the power set of $\mathrm{Hom}(U,B)$; hence a set).

Thus, assume $A_1 \overset{i_1}{\hookrightarrow} B$ and $A_2 \overset{i_2}{\hookrightarrow} B$ are such that there is a bijection

$$\begin{tikzcd} \mathrm{Hom}(U,A_1) \arrow[r, "i_1\circ -"] \arrow[d, "\rho"'] & \mathrm{Hom}(U,B) \\ \mathrm{Hom}(U,A_2) \arrow[ur, "i_2\circ -"'] & \end{tikzcd}$$

(T.S.) $A_1 \simeq A_2$.

Consider the pull-back diagram

$$\begin{tikzcd} & A_1 \arrow[dr, hook, "i_1"] & \\ A_3 \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & B \\ & A_2 \arrow[ur, hook, "i_2"'] & \end{tikzcd}$$

Claim 1. j_1 and j_2 are injective.

Pf. (for j_1; j_2 is similar)

if $Z \xrightarrow{\alpha} A_3$ is st. $j_1\alpha = 0$, then

$$\begin{tikzcd} & & A_1 \arrow[dr, "i_1"] & \\ Z \arrow[urr, "0"] \arrow[r, dashed, "\exists!"', "\alpha\ \&\ 0"] \arrow[drr, "0"'] & A_3 \arrow[ur] \arrow[dr] & & B \\ & & A_2 \arrow[ur, "i_2"'] & \end{tikzcd}$$

$i_2 j_2\alpha = i_1 j_1 \alpha = 0$

i_2 : inj $\Rightarrow$ $j_2\alpha = 0$

by uniqueness of $Z \dashrightarrow A_3$, $\alpha = 0$ □

Claim 2. j_1 is an isomorphism. (similarly j_2).

Pf. If not, we can find $f: U \to A_1$ which is not of the form $j_1\bar{f}$ for any $\bar{f}: U \to A_3$. But there is $\rho(f): U \to A_2$ st. $i_2\rho(f) = i_1 f$ $\Rightarrow$ by defn. of pull-back that $f = j_1\bar{f} = j_2\bar{f}$ for a unique $\bar{f}$. Contradiction! □

(A.3) Baer's Criterion for injectivity.–

Prop. Again $\mathcal{A}$ is abelian, (AB5) and has a generator $\mathcal{U}$.
$M \in \mathcal{A}$ is injective $\iff$ $\forall\ V \overset{v}{\hookrightarrow} \mathcal{U}$, and $\begin{array}{c} V \\ \downarrow f \\ M \end{array}$, $\exists$ a lift $\tilde{f} : \mathcal{U} \to M$ s.t. $\tilde{f} v = f$.

Pf $\Rightarrow$ obvious. Conversely, let there be given an injective morphism $A \overset{i}{\hookrightarrow} B$ and an arbitrary $g : A \to M$.

$\mathcal{P}$ = set of all $\begin{array}{c} A_1 \overset{i_1}{\hookrightarrow} B \\ \downarrow g_1 \\ M \end{array}$ s.t.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "g"'] & A_1 \arrow[dl, "g_1"] \\ M & \end{tikzcd}$$

commutes

by lemma A.2.

and

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & B \\ A_1 \arrow[ur, hook] & \end{tikzcd}$$

commutes

(up to iso. of course)

Given a chain in $\mathcal{P}$ (which is partially ordered by extn.)

$$\begin{tikzcd} A_1 \arrow[r, hook] \arrow[dr, hook, "i_1"'] & A_2 \arrow[r, hook] \arrow[d, hook, "i_2"] & A_3 \arrow[r, hook] \arrow[dl, hook, "i_3"] & \cdots \\ & B & & \end{tikzcd}$$

$\cdots$ we get

$$\begin{tikzcd} \tilde{A} = \varinjlim_{k \geq 1} A_k \arrow[r, hook, "\tilde{i}"] \arrow[d, "\tilde{g} = \varinjlim g_i"'] & B \\ M & \end{tikzcd}$$

still injective by (AB5)

Now let
$$\begin{tikzcd} C \arrow[r, hook, "j"] \arrow[d, "\tilde{g}"'] & B \\ M \end{tikzcd}$$
be a max'l elt from $\mathcal{P}$.

Claim: $C \xhookrightarrow{j} B$ is an iso.

Pf. If not, we can find $\alpha : \mathcal{U} \to B$ which cannot be written as $j \circ \bar{\alpha}$. Let $\tilde{C} = \mathrm{Image}(C \oplus \mathcal{U} \xrightarrow{j \oplus \alpha} B)$

Pictorially
$$\begin{tikzcd}
 & C \arrow[dl, "\text{natural}"'] \arrow[dr, hook, "i_c"] \arrow[drr, hook, "j"] & & \\
C \oplus \mathcal{U} \arrow[rr, "\text{defn. of image}"] & & \tilde{C} \arrow[r, hook, "\tilde{j}\text{: defn. of image}"] & B \\
 & \mathcal{U} \arrow[ul, "\text{natural}"] \arrow[ur, "i_u"'] \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

$\tilde{j}\, i_c = j$ | j & $\tilde{j}$: injective

$\Rightarrow i_c$ is inj

Again pull-back
$$\begin{tikzcd}
 & C \arrow[dr, hook, "i_c"] & \\
V \arrow[ur] \arrow[dr] & & \tilde{C} \\
 & \mathcal{U} \arrow[ur, "i_u"'] &
\end{tikzcd}$$

Exercise: $V \to \mathcal{U}$ is injective (because i_c is) (Hint: see page 5)

So
$$\begin{tikzcd}
V \arrow[r, hook] \arrow[d] & \mathcal{U} \arrow[ddl, dashed, "\text{by assumption}"] \\
C \arrow[d, "\tilde{g}"'] & \\
M &
\end{tikzcd}$$

so
$$\begin{tikzcd}
 & C \arrow[r] \arrow[rr, bend left, "\tilde{g}"] & \tilde{C} \arrow[r, dashed, "\exists !"] & M \\
V \arrow[ur] \arrow[dr] & & & \\
 & \mathcal{U} \arrow[ur] \arrow[uurr] & &
\end{tikzcd}$$

(see Lemma 12.4 page 4)

Thus by maximality of $\overset{\cancel{\sim}}{C}$, i_c is an iso.

$\Rightarrow \alpha = j\, i_c^{-1}\, i_u$ contradiction! $\square$