MID TERM 3

Problem 1. (10 points) What is the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

Problem 2. (20 points) Consider the function $f(z) = \dfrac{z}{\sin(z)}$.

(1) Prove that $z = 0$ is a removable singularity of $f(z)$.

Now, let $\dfrac{z}{\sin(z)} = \displaystyle\sum_{n=0}^{\infty} c_n z^n$ *be the Taylor series expansion of* $f(z)$ *centered at* 0.

(2) What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} c_n z^n$?

(3) Prove that $c_{2k+1} = 0$ for every $k \geq 0$.

(4) Compute c_0, c_2 and c_4.

Problem 3. (10 points) Compute $\operatorname*{Res}_{z=0} \left(\dfrac{e^{3z} - 3e^z + 2}{z^4} \right)$.

Problem 4. (10 points) Let C be the counterclockwise circle of radius 3, centered at 0. Prove that $\displaystyle\int_C \frac{2z^3 + 1}{z^2(z-1)(z-2)}\, dz = 4\pi \mathbf{i}$.

Problem 5. (10 points) Compute the Laurent series expansion of $\dfrac{1}{z^2(z-2)}$ near $z = 2$.

Problem 6. (20 points) Prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2 + \cos(\theta))^2} = \frac{4\pi}{3\sqrt{3}}$.

Problem 7. (20 points) Prove that $\displaystyle\int_0^{\infty} \frac{x \sin(x)}{x^2 + 1}\, dx = \frac{\pi}{2e}$.

Bonus. Let $a \in \mathbb{R}_{>0}$ be a fixed positive real number. Prove that $\displaystyle\int_0^{\pi} \tan(x + a\mathbf{i})\, dx = \pi \mathbf{i}$.