COMPLEX ANALYSIS: MID TERM 1

Name: Solutions. email: ____________

Graduating student (y/n): ________.

Problem 1. Sketch the following domain and state whether it is open/closed/neither, connected or not, bounded or not (**no justification needed**).

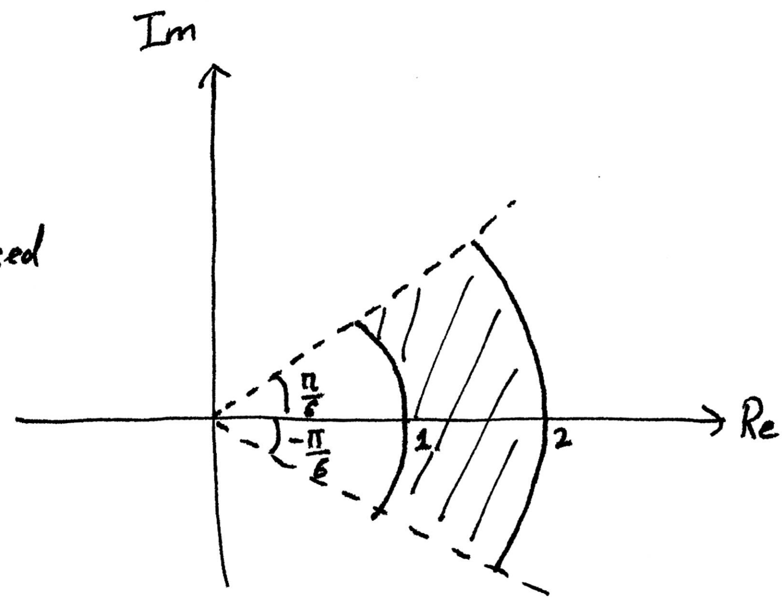
$$S = \left\{ z \in \mathbb{C} \text{ such that } -\frac{\pi}{6} < \arg(z) < \frac{\pi}{6} \text{ and } 1 \leq |z| \leq 2 \right\}$$

This domain is

(1) Neither open nor closed

(2) Connected

(3) Bounded



Problem 2. Compute the following limits (simplify your answer to the form $x + yi$).

(a) $\lim_{z \to i} \frac{z^3+3}{z+1}$ $= \frac{i^3+3}{i+1} = \frac{3-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{2-4i}{2}$

$= 1-2i$

(b) $\lim_{z \to 0} \frac{e^{\sin(z)} - 1 - z}{z^2}$ $= \lim_{z \to 0} \frac{e^{\sin(z)} \cdot \cos(z) - 1}{2z}$

$= \lim_{z \to 0} \frac{e^{\sin(z)} \cdot \cos^2(z) - e^{\sin(z)} \cdot \sin(z)}{2}$

$= \frac{e^0 \cdot 1 - 0}{2} = \frac{1}{2}.$

Problem 3. Let $u(x, y) = xe^x \cos(y) - e^x y \sin(y)$.

(a) Verify that $u(x, y)$ is a harmonic function.

$u_x = (xe^x + e^x)\cos(y) - e^x y \sin(y)$

(1) $u_{xx} = (xe^x + 2e^x)\cos(y) - e^x y \sin(y)$

$u_y = -xe^x \sin(y) - e^x(\sin(y) + y\cos(y))$

(2) $u_{yy} = -xe^x \cos(y) - e^x(2\cos(y) - y\sin y)$

(1) + (2) : $u_{xx} + u_{yy} = 0$ Hence, $u(x, y)$ is harmonic.

(b) Find $v(x,y)$ so that $u(x,y)+\mathbf{i}v(x,y)$ is a $\mathbb{C}$–differentiable function.
Useful information: Antiderivative of xe^x *is* $xe^x - e^x$*; and antiderivative of* $y\sin(y)$ *is* $-y\cos(y)+\sin(y)$.

$$V_x = -U_y = xe^x \sin(y) + e^x(\sin(y) + y\cos(y))$$

$$\Rightarrow v = (xe^x - e^x)\sin(y) + e^x(\sin(y) + y\cos(y)) + f(y)$$

$$= xe^x \sin(y) + e^x y\cos(y) + f(y)$$

Substitute in $v_y = u_x$ to get:

$$xe^x\cos(y) + e^x(\cos(y) - y\sin(y)) + f'(y)$$

$$= xe^x\cos(y) + e^x\cos(y) - e^x y\sin(y)$$

$\Rightarrow f'(y) = 0$. Hence $f(y) = C \in \mathbb{R}$, and

$$v(x,y) = xe^x\sin(y) + e^x y\cos(y) + C.$$

Problem 4. Let $f(z) = (3z-\mathbf{i})^\pi$. On the complex plane, indicate where $f(z)$ is not defined? Compute $f'(z)$.

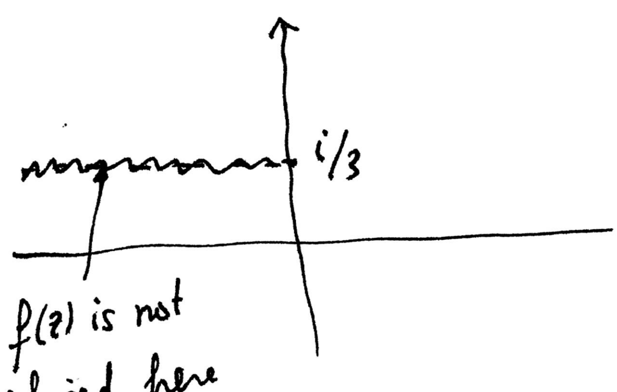
$f(z) = e^{\pi \ln(3z-i)}$ is not defined for z such that

$$3z - i \in \mathbb{R}_{\leq 0}$$

i.e. $3z \in \mathbb{R}_{\leq 0} + i$

$z \in \mathbb{R}_{\leq 0} + \frac{i}{3}$

$$f'(z) = \pi(3z-i)^{\pi-1} \cdot 3$$

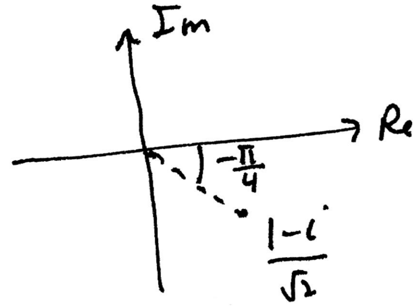


Problem 5. Compute the real and imaginary parts of the following complex numbers.

(a) $\left(\frac{1-\mathrm{i}}{\sqrt{2}}\right)^{1+\mathrm{i}}$. $= e^{(1+i)\left(\ln\left(\frac{1-i}{\sqrt{2}}\right)\right)}$

$= e^{(1+i)\left(\cancel{\ln(1)}^{\,0} - \frac{\pi}{4}i\right)}$

$= e^{-\frac{\pi}{4}i} \cdot e^{\pi/4}$



$\left|\frac{1-i}{\sqrt{2}}\right| = 1$

Real part $= e^{\pi/4} \cdot \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} e^{\pi/4}$

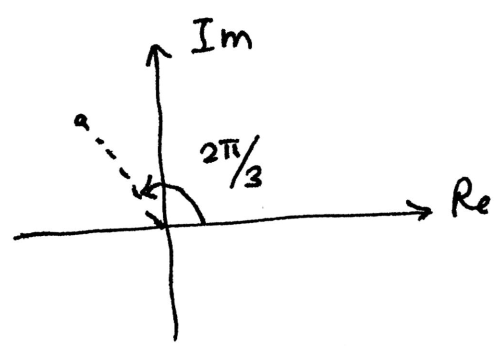
Imaginary part $= e^{\pi/4} \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} e^{\pi/4}$

(b) $\left(-1+\mathrm{i}\sqrt{3}\right)^{6}$. $-1 + i\sqrt{3} = 2 \cdot e^{2\pi/3\, i}$

$\Rightarrow (-1+i\sqrt{3})^6 = 2^6 \cdot e^{\frac{2\pi}{3} \cdot 6 i}$

$= 64 \cdot e^{4\pi i}$

$= 64.$



Problem 6. Write the definition of $\cos(z), \sin(z)$ for $z \in \mathbb{C}$. Use this to verify the following identity:

$$\sin(z+w)\sin(z-w) = \frac{\cos(2w) - \cos(2z)}{2}.$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z+w)\sin(z-w) = \frac{\left(e^{i(z+w)} - e^{-i(z+w)}\right)\left(e^{i(z-w)} - e^{-i(z-w)}\right)}{-4}$$

$$= \frac{e^{2iz} - e^{2iw} - e^{-2iw} + e^{-2iz}}{-4} = \frac{1}{2}\left(\frac{e^{2iw} + e^{-2iw}}{2} - \frac{e^{2iz} + e^{-2iz}}{2}\right)$$

$$= \frac{1}{2}\left(\cos(2w) - \cos(2z)\right)$$

Problem 7. Compute all complex numbers z such that $z^5 + 32 = 0$. Compute the sum of all these numbers.

$$z^5 = -32 = 32 \cdot e^{\pi i}$$

$$= 2^5 \cdot e^{\pi i}$$

$z_0 = 2 \cdot e^{\frac{\pi}{5} i}$. Let $w = e^{\frac{2\pi i}{5}}$. Then other 4 solns are:

$z_1 = 2 \cdot e^{(\pi/5)i} \cdot e^{\frac{2\pi i}{5}} = z_0 w$; $z_2 = z_0 w^2$; $z_3 = z_0 w^3$; $z_4 = z_0 w^4$

$$\text{Sum} = z_0\left(1 + w + w^2 + w^3 + w^4\right)$$

$$= z_0 \frac{w^5 - 1}{w - 1} = z_0 \frac{1-1}{w-1} = 0.$$

$(w \neq 1)$