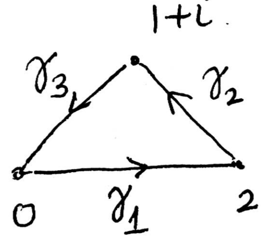
COMPLEX ANALYSIS: MID TERM 2

Name: ______________________ email: ______________________

Problem 1. (10 points) Let C be the counterclockwise oriented triangle with vertices 0, 2 and $1 + \mathbf{i}$.

(1) Write a parametrization for each of the three smooth parts of C.

- $\gamma_1 : [0,1] \to \mathbb{C}$; $\gamma_1(t) = 2t$.
- $\gamma_2 : [0,1] \to \mathbb{C}$; $\gamma_2(t) = (2-t) + ti$
- $\gamma_3 : [0,1] \to \mathbb{C}$; $\gamma_3(t) = (1+i)(1-t)$



(2) Compute $\int_C \mathrm{Re}(z)^2\, dz$.

$$\int_{\gamma_1} \mathrm{Re}(z)^2 dz = \int_0^1 4t^2 \cdot 2 \cdot dt = \left[8\frac{t^3}{3}\right]_0^1 = \frac{8}{3}$$

$$\int_{\gamma_2} \mathrm{Re}(z)^2 dz = \int_0^1 (2-t)^2 \cdot (-1+i)\, dt = (-1+i)\left[\frac{(t-2)^3}{3}\right]_0^1$$

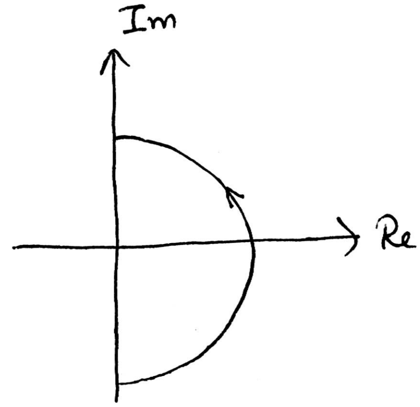
$$= (-1+i)\left(-\frac{1}{3} + \frac{8}{3}\right)$$

$$\int_{\gamma_3} \mathrm{Re}(z)^2 dz = \int_0^1 (1-t)^2 (-1-i)\, dt = (-1-i)\left[\frac{(t-1)^3}{3}\right]_0^1 = (-1-i)\left(\frac{1}{3}\right)$$

$$\Rightarrow \int_C \mathrm{Re}(z)^2 dz = \frac{8}{3} + \frac{7}{3}(-1+i) + \frac{1}{3}(-1-i) = \left(\frac{7}{3} - \frac{1}{3}\right)i = 2i.$$

Problem 2. **(10 points)** Let R be a positive real number and consider $\gamma_R : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{C}$, given by: $\gamma_R(t) = Re^{it}$. Prove that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-z}}{z^3}\, dz = 0.$$



For z on γ_R, $Re(z) \geq 0$.

$$\Rightarrow |e^{-z}| = e^{-Re(z)} \leq 1.$$

$$\left| \int_{\gamma_R} \frac{e^{-z}}{z^3}\, dz \right| \leq \frac{1}{R^3} \cdot \pi R = \frac{\pi}{R^2} \to 0 \text{ as } R\to\infty$$

$$\Rightarrow \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-z}}{z^3}\, dz = 0.$$

Problem 3. (10 points)

(1) Compute an antiderivative of $z \cdot e^z$.

This antiderivative can be computed using integration by parts

$$\int z e^z\, dz = z \cdot e^z - \int 1 \cdot e^z\, dz$$

$$= z e^z - e^z$$

[Verification: $\frac{d}{dz}\left(z \cdot e^z - e^z\right) = z \cdot e^z + \cancel{1 \cdot e^z} - \cancel{e^z}$

$= z \cdot e^z$. ✓]

Problem 5. (10 points) Compute the partial fraction decomposition of $\dfrac{z^3 - z + 1}{z^2(z-1)(z-\mathbf{i})}$.

$$\frac{z^3 - z + 1}{z^2(z-1)(z-i)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} + \frac{D}{z-i}$$

Clearing denominator, we get:

$$z^3 - z + 1 = A \cdot z \cdot (z-1)(z-i) + B \cdot (z-1)(z-i) + C \cdot z^2 (z-i) + D \cdot z^2 (z-1)$$

Set $z=0$, to get $1 = B \cdot (-1)(-i) \Rightarrow \boxed{B = \frac{1}{i} = -i}$

Set $z=1$, to get $1 = C \cdot (1-i)$ $\quad \boxed{C = \frac{1}{1-i} \left(= \frac{1+i}{2}\right)}$

Set $z=i$, to get $1-2i = D \cdot i^2 \cdot (i-1)$

So, $\boxed{D = \frac{1-2i}{1-i}}$. Finally $A + C + D = 1$ (coeff of z^3)

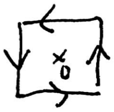
$$\boxed{A = 1 - \left(\frac{1}{1-i} + \frac{1-2i}{1-i}\right) = -1}$$

Problem 6. (10 points) Let C be the counterclockwise oriented square with vertices $-1-\mathbf{i}, 1-\mathbf{i}, 1+\mathbf{i}, -1+\mathbf{i}$. Let $a \in \mathbb{C}$ be a fixed complex number, and $n \in \mathbb{Z}_{\geq 0}$ be a non–negative integer. Prove that

$$\frac{1}{2\pi\mathbf{i}} \int_C \frac{e^{az}}{z^{n+1}}\, dz = \frac{a^n}{n!}.$$

By Cauchy Integral Formula

$$\frac{1}{2\pi i} \int \frac{e^{az}}{z^{n+1}}\, dz = \frac{1}{n!} \left[\frac{d^n}{dz^n} \left(e^{az}\right) \right]_{\text{set } z=0}$$



$$= \frac{1}{n!} \left[a^n \cdot e^{az} \right]_{z=0} = \frac{a^n}{n!}.$$

(2) Let $\gamma : [0,1] \to \mathbb{C}$ be given by: $\gamma(t) = 2t + t(t-1)\mathbf{i}$. Compute $\int_\gamma z \cdot e^z \, dz$.

$\gamma(0) = 0$ and $\gamma(1) = 2$.

$$\int_\gamma z \cdot e^z \cdot dz = \left[z \cdot e^z - e^z \right]_0^2 = \left(2 \cdot e^2 - e^2\right) - \left(0 \cdot e^0 - e^0\right)$$

$$= e^2 + 1.$$

Problem 4. (10 points) Let C be a counterclockwise contour such that $1 \in \text{Interior}(C)$ and $0 \in \text{Exterior}(C)$. Compute $\dfrac{1}{2\pi\mathbf{i}} \displaystyle\int_C \frac{e^z}{z(z-1)^3} \, dz$.

By Cauchy's integral formula

$$\frac{1}{2\pi i} \int_C \frac{e^z}{z(z-1)^3} \, dz = \frac{1}{2!} \left(\frac{d^2}{dz^2} \left(\frac{e^z}{z} \right) \right)_{\text{set } z = 1}$$

$$\left(z^{-1} e^z\right)' = -z^{-2} e^z + z^{-1} e^z$$

$$\left(z^{-1} e^z\right)'' = +2 z^{-3} e^z - 2 z^{-2} e^z + z^{-1} e^z$$

$$\Rightarrow \frac{1}{2\pi i} \int_C \frac{e^z}{z(z-1)^3} \, dz = \frac{1}{2} \left(2 z^{-3} e^z - 2 z^{-2} e^z + z^{-1} e^z \right)_{\text{set } z=1}$$

$$= \frac{1}{2} \left(\cancel{2e^1} - \cancel{2e^1} + e^1 \right) = \frac{e}{2}.$$